(4,2) - Association Type Identities

Taking the 10 identities with $r=1$, for each of them we can replace the element in one of the 4 positions by another one. I.e. there are 40 identities of rank 2 all in all.
These are also given in [1] and to facilitate comparisons, we'll list the labelling $H_{\dots}$ used there as well.

Identity	Abbreviation	[1]	Type of Quasigroup
$((\mathbf{BA})\mathbf{A})\mathbf{A}=\mathbf{B}((\mathbf{AA})\mathbf{A})$	$120XXX$	H_8	
$((\mathbf{AB})\mathbf{A})\mathbf{A}=\mathbf{A}((\mathbf{BA})\mathbf{A})$	$12X0XX$	H_6	
$((\mathbf{AA})\mathbf{B})\mathbf{A}=\mathbf{A}((\mathbf{AB})\mathbf{A})$	$12XX0X$	H_7	
$((\mathbf{AA})\mathbf{A})\mathbf{B}=\mathbf{A}((\mathbf{AA})\mathbf{B})$	$12XXX0$	H_5	
$((\mathbf{BA})\mathbf{A})\mathbf{A}=(\mathbf{BA})(\mathbf{AA})$ $[\mathbf{B}',\mathbf{A},\mathbf{A}]$	$130XXX$	H_{21}	**Right-alternative***
$((\mathbf{AB})\mathbf{A})\mathbf{A}=(\mathbf{AB})(\mathbf{AA})$	$13X0XX$	H_{19}	**Right-alternative***
$((\mathbf{AA})\mathbf{B})\mathbf{A}=(\mathbf{AA})(\mathbf{BA})$	$13XX0X$	H_{20}	3^{rd} **Jordan identity**
$((\mathbf{AA})\mathbf{A})\mathbf{B}=(\mathbf{AA})(\mathbf{AB})$	$13XXX0$	H_{11}	
$((\mathbf{BA})\mathbf{A})\mathbf{A}=(\mathbf{B}(\mathbf{AA}))\mathbf{A}$	$140XXX$		Trivial if **right-alternative**
$((\mathbf{AB})\mathbf{A})\mathbf{A}=(\mathbf{A}(\mathbf{BA}))\mathbf{A}$	$14X0XX$		Trivial if **flexible**
$((\mathbf{AA})\mathbf{B})\mathbf{A}=(\mathbf{A}(\mathbf{AB}))\mathbf{A}$	$14XX0X$		Trivial if **left-alternative**
$((\mathbf{AA})\mathbf{A})\mathbf{B}=(\mathbf{A}(\mathbf{AA}))\mathbf{B}$	$14XXX0$		Trivial if **mono-associative**
$((\mathbf{BA})\mathbf{A})\mathbf{A}=\mathbf{B}(\mathbf{A}(\mathbf{AA}))$	$150XXX$	H_8	
$((\mathbf{AB})\mathbf{A})\mathbf{A}=\mathbf{A}(\mathbf{B}(\mathbf{AA}))$	$15X0XX$	H_2	
$((\mathbf{AA})\mathbf{B})\mathbf{A}=\mathbf{A}(\mathbf{A}(\mathbf{BA}))$	$15XX0X$	H_3	
$((\mathbf{AA})\mathbf{A})\mathbf{B}=\mathbf{A}(\mathbf{A}(\mathbf{AB}))$	$15XXX0$	H_1	
$\mathbf{B}((\mathbf{AA})\mathbf{A})=(\mathbf{BA})(\mathbf{AA})$	$230XXX$	H_{18}	
$\mathbf{A}((\mathbf{BA})\mathbf{A})=(\mathbf{AB})(\mathbf{AA})$	$23X0XX$	H_{16}	
$\mathbf{A}((\mathbf{AB})\mathbf{A})=(\mathbf{AA})(\mathbf{BA})$	$23XX0X$	H_{17}	
$\mathbf{A}((\mathbf{AA})\mathbf{B})=(\mathbf{AA})(\mathbf{AB})$	$23XXX0$	H_{15}	4^{th} **Jordan identity**
$\mathbf{B}((\mathbf{AA})\mathbf{A})=(\mathbf{B}(\mathbf{AA}))\mathbf{A}$	$240XXX$	H_4	
$\mathbf{A}((\mathbf{BA})\mathbf{A})=(\mathbf{A}(\mathbf{BA}))\mathbf{A}$	$24X0XX$	H_{25}	
$\mathbf{A}((\mathbf{AB})\mathbf{A})=(\mathbf{A}(\mathbf{AB}))\mathbf{A}$	$24XX0X$	H_{26}	
$\mathbf{A}((\mathbf{AA})\mathbf{B})=(\mathbf{A}(\mathbf{AA}))\mathbf{B}$	$24XXX0$	H_5	
$\mathbf{B}((\mathbf{AA})\mathbf{A})=(\mathbf{B}(\mathbf{A}(\mathbf{AA}))$	$250XXX$		Trivial if **mono-associative**
$\mathbf{A}((\mathbf{BA})\mathbf{A})=(\mathbf{A}(\mathbf{B}(\mathbf{AA}))$	$25X0XX$		Trivial if **right-alternative**
$\mathbf{A}((\mathbf{AB})\mathbf{A})=(\mathbf{A}(\mathbf{A}(\mathbf{BA}))$	$25XX0X$		Trivial if **flexible**
$\mathbf{A}((\mathbf{AA})\mathbf{B})=(\mathbf{A}(\mathbf{A}(\mathbf{AB}))$	$25XXX0$		Trivial if **left-alternative**
$(\mathbf{BA})(\mathbf{AA})=(\mathbf{B}(\mathbf{AA}))\mathbf{A}$	$340XXX$	H_{14}	1^{st} **Jordan identity**
$(\mathbf{AB})(\mathbf{AA})=(\mathbf{A}(\mathbf{BA}))\mathbf{A}$	$34X0XX$	H_{12}	
$(\mathbf{AA})(\mathbf{BA})=(\mathbf{A}(\mathbf{AB}))\mathbf{A}$	$34XX0X$	H_{13}	
$(\mathbf{AA})(\mathbf{AB})=(\mathbf{A}(\mathbf{AA}))\mathbf{B}$	$34XXX0$	H_{11}	
$(\mathbf{BA})(\mathbf{AA})=\mathbf{B}(\mathbf{A}(\mathbf{AA}))$	$350XXX$	H_{18}	
$(\mathbf{AB})(\mathbf{AA})=\mathbf{A}(\mathbf{B}(\mathbf{AA}))$	$35X0XX$	H_{23}	2^{nd} **Jordan identity**
$(\mathbf{AA})(\mathbf{BA})=\mathbf{A}(\mathbf{A}(\mathbf{BA}))$	$35XX0X$	H_{24}	
$(\mathbf{AA})(\mathbf{AB})=\mathbf{A}(\mathbf{A}(\mathbf{AB}))$	$35XXX0$	H_{22}	
$(\mathbf{B}(\mathbf{AA}))\mathbf{A}=\mathbf{B}(\mathbf{A}(\mathbf{AA}))$	$450XXX$	H_4	
$(\mathbf{A}(\mathbf{BA}))\mathbf{A}=\mathbf{A}(\mathbf{B}(\mathbf{AA}))$	$45X0XX$	H_9	
$(\mathbf{A}(\mathbf{AB}))\mathbf{A}=\mathbf{A}(\mathbf{A}(\mathbf{BA}))$	$45XX0X$	H_{10}	
$(\mathbf{A}(\mathbf{AA}))\mathbf{B}=\mathbf{A}(\mathbf{A}(\mathbf{AB}))$	$45XXX0$	H_1	

* Due to the property of unique resolvability of quasigroups.

Furthermore we can replace two elements by two other identical elements. For each identity there are 3 inequivalent ways of doing so.
Hence we get another 30 rank-2 identities, listed in the following table:

Identity	Abbreviation	[1]	Type of Quasigroup
$((\mathbf{AA})\mathbf{B})\mathbf{B}=\mathbf{A}((\mathbf{AB})\mathbf{B})$	$12XX00$	H_{48}	
$((\mathbf{AB})\mathbf{A})\mathbf{B}=\mathbf{A}((\mathbf{BA})\mathbf{B})$	$12X0X0$	H_{33}	
$((\mathbf{AB})\mathbf{B})\mathbf{A}=\mathbf{A}((\mathbf{BB})\mathbf{A})$	$12X00X$	H_{39}	
$((\mathbf{AA})\mathbf{B})\mathbf{B}=(\mathbf{AA})(\mathbf{BB})$	$13XX00$	H_{43}	
$((\mathbf{AB})\mathbf{A})\mathbf{B}=(\mathbf{AB})(\mathbf{AB})$	$13X0X0$	H_{29}	
$((\mathbf{AB})\mathbf{B})\mathbf{A}=(\mathbf{AB})(\mathbf{BA})$	$13X00X$	H_{37}	
$((\mathbf{AA})\mathbf{B})\mathbf{B}=(\mathbf{A}(\mathbf{AB}))\mathbf{B}$	$14XX00$		Trivial if **left-alternative**
$((\mathbf{AB})\mathbf{A})\mathbf{B}=(\mathbf{A}(\mathbf{BA}))\mathbf{B}$	$14X0X0$		Trivial if **flexible**

$((\mathbf{AB})\mathbf{B})\mathbf{A} = (\mathbf{A}(\mathbf{BB}))\mathbf{A}$	$14X00X$		Trivial if **right-alternative**
$((\mathbf{AA})\mathbf{B})\mathbf{B} = \mathbf{A}(\mathbf{A}(\mathbf{BB}))$	$15XX00$	H_{47}	
$((\mathbf{AB})\mathbf{A})\mathbf{B} = \mathbf{A}(\mathbf{B}(\mathbf{AB}))$	$15X0X0$	H_{31}	
$((\mathbf{AB})\mathbf{B})\mathbf{A} = \mathbf{A}(\mathbf{B}(\mathbf{BA}))$	$15X00X$	H_{41}	
$\mathbf{A}((\mathbf{AB})\mathbf{B}) = (\mathbf{AA})(\mathbf{BB})$	$23XX00$	H_{46}	
$\mathbf{A}((\mathbf{BA})\mathbf{B}) = (\mathbf{AB})(\mathbf{AB})$	$23X0X0$	H_{28}	
$\mathbf{A}((\mathbf{BB})\mathbf{A}) = (\mathbf{AB})(\mathbf{BA})$	$23X00X$	H_{35}	
$\mathbf{A}((\mathbf{AB})\mathbf{B}) = (\mathbf{A}(\mathbf{AB}))\mathbf{B}$	$24XX00$	H_{50}	
$\mathbf{A}((\mathbf{BA})\mathbf{B}) = (\mathbf{A}(\mathbf{BA}))\mathbf{B}$	$24X0X0$	H_{34}	
$\mathbf{A}((\mathbf{BB})\mathbf{A}) = (\mathbf{A}(\mathbf{BB}))\mathbf{A}$	$24X00X$	H_{40}	
$\mathbf{A}((\mathbf{AB})\mathbf{B}) = \mathbf{A}(\mathbf{A}(\mathbf{BB}))$	$25XX00$		Trivial if **right-alternative**
$\mathbf{A}((\mathbf{BA})\mathbf{B}) = \mathbf{A}(\mathbf{B}(\mathbf{AB}))$	$25X0X0$		Trivial if **flexible**
$\mathbf{A}((\mathbf{BB})\mathbf{A}) = \mathbf{A}(\mathbf{B}(\mathbf{BA}))$	$25X00X$		Trivial if **left-alternative**
$(\mathbf{AA})(\mathbf{BB}) = (\mathbf{A}(\mathbf{AB}))\mathbf{B}$	$34XX00$	H_{44}	
$(\mathbf{AB})(\mathbf{AB}) = (\mathbf{A}(\mathbf{BA}))\mathbf{B}$	$34X0X0$	H_{30}	
$(\mathbf{AB})(\mathbf{BA}) = (\mathbf{A}(\mathbf{BB}))\mathbf{A}$	$34X00X$	H_{38}	
$(\mathbf{AA})(\mathbf{BB}) = \mathbf{A}(\mathbf{A}(\mathbf{BB}))$	$35XX00$	H_{45}	
$(\mathbf{AB})(\mathbf{AB}) = \mathbf{A}(\mathbf{B}(\mathbf{AB}))$	$35X0X0$	H_{27}	
$(\mathbf{AB})(\mathbf{BA}) = \mathbf{A}(\mathbf{B}(\mathbf{BA}))$	$35X00X$	H_{36}	
$(\mathbf{A}(\mathbf{AB}))\mathbf{B} = \mathbf{A}(\mathbf{A}(\mathbf{BB}))$	$45XX00$	H_{49}	
$(\mathbf{A}(\mathbf{BA}))\mathbf{B} = \mathbf{A}(\mathbf{B}(\mathbf{AB}))$	$45X0X0$	H_{32}	
$(\mathbf{A}(\mathbf{BB}))\mathbf{A} = \mathbf{A}(\mathbf{B}(\mathbf{BA}))$	$45X00X$	H_{42}	

See also:

- **Association type identities**
- **(4,3) - association type identities**

Google books:

- [1] Geometry and Algebra of Multidimensional Three-webs (1992) - M. A. Akivis, A. M. Shelekhov local bct. 64 - brl. 10

(4,3) - Association Type Identities

There are $42 = 6$ possibilities to place 2 identical elements in a 4-tuple. Thus one gets $6 \cdot 10 = 60$ rank-3 identities, listed in the following table:

Identity	Abbreviation	Type of Quasigroup
$((\mathbf{AA})\mathbf{B})\mathbf{C} = \mathbf{A}((\mathbf{AB})\mathbf{C})$	$12XX00$	RG1
$((\mathbf{AB})\mathbf{A})\mathbf{C} = \mathbf{A}((\mathbf{BA})\mathbf{C})$	$12X0X0$	RG3
$((\mathbf{AB})\mathbf{C})\mathbf{A} = \mathbf{A}((\mathbf{BC})\mathbf{A})$	$12X00X$	
$((\mathbf{AB})\mathbf{B})\mathbf{C} = \mathbf{A}((\mathbf{BB})\mathbf{C})$	$120XX0$	RC4
$((\mathbf{AB})\mathbf{C})\mathbf{B} = \mathbf{A}((\mathbf{BC})\mathbf{B})$	$120X0X$	**Right Bol**
$((\mathbf{AB})\mathbf{C})\mathbf{C} = \mathbf{A}((\mathbf{BC})\mathbf{C})$	$1200XX$	RC2
$((\mathbf{AA})\mathbf{B})\mathbf{C} = (\mathbf{AA})(\mathbf{BC})$	$13XX00$	Left Nuclear Square
$((\mathbf{AB})\mathbf{A})\mathbf{C} = (\mathbf{AB})(\mathbf{AC})$	$13X0X0$	**Group***
$((\mathbf{AB})\mathbf{C})\mathbf{A} = (\mathbf{AB})(\mathbf{CA})$	$13X00X$	**Group***
$((\mathbf{AB})\mathbf{B})\mathbf{C} = (\mathbf{AB})(\mathbf{BC})$	$130XX0$	**Group***
$((\mathbf{AB})\mathbf{C})\mathbf{B} = (\mathbf{AB})(\mathbf{CB})$	$130X0X$	**Group***
$((\mathbf{AB})\mathbf{C})\mathbf{C} = (\mathbf{AB})(\mathbf{CC})$ $\Leftrightarrow [\mathbf{A}', \mathbf{C}, \mathbf{C}] = 0$	$1300XX$	**Right Alternative***
$((\mathbf{AA})\mathbf{B})\mathbf{C} = (\mathbf{A}(\mathbf{AB}))\mathbf{C}$	$14XX00$	**Left Alternative**

$((\mathbf{AB})\mathbf{A})\mathbf{C} = (\mathbf{A}(\mathbf{BA}))\mathbf{C}$	$14X0X0$	**Flexible**
$((\mathbf{AB})\mathbf{C})\mathbf{A} = (\mathbf{A}(\mathbf{BC}))\mathbf{A}$	$14X00X$	**Group**
$((\mathbf{AB})\mathbf{B})\mathbf{C} = (\mathbf{A}(\mathbf{BB}))\mathbf{C}$	$140XX0$	**Right Alternative**
$((\mathbf{AB})\mathbf{C})\mathbf{B} = (\mathbf{A}(\mathbf{BC}))\mathbf{B}$	$140X0X$	**Group**
$((\mathbf{AB})\mathbf{C})\mathbf{C} = (\mathbf{A}(\mathbf{BC}))\mathbf{C}$	$1400XX$	**Group**
$((\mathbf{AA})\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{A}(\mathbf{BC}))$	$15XX00$	LC3
$((\mathbf{AB})\mathbf{A})\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{AC}))$	$15X0X0$	**Left Moufang**
$((\mathbf{AB})\mathbf{C})\mathbf{A} = \mathbf{A}(\mathbf{B}(\mathbf{CA}))$	$15X00X$	Middle Extra
$((\mathbf{AB})\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{BC}))$	$150XX0$	*C*
$((\mathbf{AB})\mathbf{C})\mathbf{B} = \mathbf{A}(\mathbf{B}(\mathbf{CB}))$	$150X0X$	**Right Moufang**
$((\mathbf{AB})\mathbf{C})\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{CC}))$	$1500XX$	RC3
$\mathbf{A}((\mathbf{AB})\mathbf{C}) = (\mathbf{AA})(\mathbf{BC})$	$23XX00$	
$\mathbf{A}((\mathbf{BA})\mathbf{C}) = (\mathbf{AB})(\mathbf{AC})$	$23X0X0$	Left Extra
$\mathbf{A}((\mathbf{BC})\mathbf{A}) = (\mathbf{AB})(\mathbf{CA})$	$23X00X$	**Middle Moufang**
$\mathbf{A}((\mathbf{BB})\mathbf{C}) = (\mathbf{AB})(\mathbf{BC})$	$230XX0$	
$\mathbf{A}((\mathbf{BC})\mathbf{B}) = (\mathbf{AB})(\mathbf{CB})$	$230X0X$	
$\mathbf{A}((\mathbf{BC})\mathbf{C}) = (\mathbf{AB})(\mathbf{CC})$	$2300XX$	RC1
$\mathbf{A}((\mathbf{AB})\mathbf{C}) = (\mathbf{A}(\mathbf{AB}))\mathbf{C}$	$24XX00$	**Group***
$\mathbf{A}((\mathbf{BA})\mathbf{C}) = (\mathbf{A}(\mathbf{BA}))\mathbf{C}$	$24X0X0$	**Group***
$\mathbf{A}((\mathbf{BC})\mathbf{A}) = (\mathbf{A}(\mathbf{BC}))\mathbf{A}$	$24X00X$	**Flexible***
$\mathbf{A}((\mathbf{BB})\mathbf{C}) = (\mathbf{A}(\mathbf{BB}))\mathbf{C}$	$240XX0$	Middle Nuclear Square
$\mathbf{A}((\mathbf{BC})\mathbf{B}) = (\mathbf{A}(\mathbf{BC}))\mathbf{B}$	$240X0X$	**Group***
$\mathbf{A}((\mathbf{BC})\mathbf{C}) = (\mathbf{A}(\mathbf{BC}))\mathbf{C}$	$2400XX$	**Group***
$\mathbf{A}((\mathbf{AB})\mathbf{C}) = \mathbf{A}(\mathbf{A}(\mathbf{BC}))$	$25XX00$	**Group**
$\mathbf{A}((\mathbf{BA})\mathbf{C}) = \mathbf{A}(\mathbf{B}(\mathbf{AC}))$	$25X0X0$	**Group**
$\mathbf{A}((\mathbf{BC})\mathbf{A}) = \mathbf{A}(\mathbf{B}(\mathbf{CA}))$	$25X00X$	**Group**
$\mathbf{A}((\mathbf{BB})\mathbf{C}) = \mathbf{A}(\mathbf{B}(\mathbf{BC}))$	$250XX0$	**Left Alternative**
$\mathbf{A}((\mathbf{BC})\mathbf{B}) = \mathbf{A}(\mathbf{B}(\mathbf{CB}))$	$250X0X$	**Flexible**
$\mathbf{A}((\mathbf{BC})\mathbf{C}) = \mathbf{A}(\mathbf{B}(\mathbf{CC}))$	$2500XX$	**Right Alternative**
$(\mathbf{AA})(\mathbf{BC}) = (\mathbf{A}(\mathbf{AB}))\mathbf{C}$	$34XX00$	LC1
$(\mathbf{AB})(\mathbf{AC}) = (\mathbf{A}(\mathbf{BA}))\mathbf{C}$	$34X0X0$	
$(\mathbf{AB})(\mathbf{CA}) = (\mathbf{A}(\mathbf{BC}))\mathbf{A}$	$34X00X$	**Middle Moufang**
$(\mathbf{AB})(\mathbf{BC}) = (\mathbf{A}(\mathbf{BB}))\mathbf{C}$	$340XX0$	
$(\mathbf{AB})(\mathbf{CB}) = (\mathbf{A}(\mathbf{BC}))\mathbf{B}$	$340X0X$	Right Extra
$(\mathbf{AB})(\mathbf{CC}) = (\mathbf{A}(\mathbf{BC}))\mathbf{C}$	$3400XX$	LG2
$(\mathbf{AA})(\mathbf{BC}) = \mathbf{A}(\mathbf{A}(\mathbf{BC}))$	$35XX00$	RG2 = **Left Alternative***
$(\mathbf{AB})(\mathbf{AC}) = \mathbf{A}(\mathbf{B}(\mathbf{AC}))$	$35X0X0$	**Group***
$(\mathbf{AB})(\mathbf{CA}) = \mathbf{A}(\mathbf{B}(\mathbf{CA}))$	$35X00X$	**Group***
$(\mathbf{AB})(\mathbf{BC}) = \mathbf{A}(\mathbf{B}(\mathbf{BC}))$	$350XX0$	**Group***
$(\mathbf{AB})(\mathbf{CB}) = \mathbf{A}(\mathbf{B}(\mathbf{CB}))$	$350X0X$	**Group***
$(\mathbf{AB})(\mathbf{CC}) = \mathbf{A}(\mathbf{B}(\mathbf{CC}))$	$3500XX$	Right Nuclear Square
$(\mathbf{A}(\mathbf{AB}))\mathbf{C} = \mathbf{A}(\mathbf{A}(\mathbf{BC}))$	$45XX00$	LC2
$(\mathbf{A}(\mathbf{BA}))\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{AC}))$	$45X0X0$	**Left Bol**
$(\mathbf{A}(\mathbf{BC}))\mathbf{A} = \mathbf{A}(\mathbf{B}(\mathbf{CA}))$	$45X00X$	
$(\mathbf{A}(\mathbf{BB}))\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{BC}))$	$450XX0$	LC4
$(\mathbf{A}(\mathbf{BC}))\mathbf{B} = \mathbf{A}(\mathbf{B}(\mathbf{CB}))$	$450X0X$	LG3
$(\mathbf{A}(\mathbf{BC}))\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{CC}))$	$4500XX$	LG1

* Due to the property of unique resolvability of quasigroups.

Hence (at maximum) 30 out of the 60 identities are non-trivial degree-4 identities.

See also:

- **Association type identities**
- **(4,3) - association type identities**

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

***-Algebra**

A ***-Algebra** is an algebra equipped with an **involution *** .

Links:

- WIKIPEDIA - *-Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Abstract Algebra

Abstract Algebra is primarily the study of specific **algebraic structures** and their properties.

See also:

- **Universal algebra**

Links:

- WIKIPEDIA - Abstract Algebra
- WIKIPEDIA - List of Abstract Algebra Topics

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Akivis Algebra

An **Akivis Algebra** is a vector space endowed with a skew-symmetric bilinear product a and a trilinear product b satisfying the identity

$$a(\mathbf{A}, a(\mathbf{B}, \mathbf{C})) + a(\mathbf{B}, a(\mathbf{C}, \mathbf{A})) + a(\mathbf{C}, a(\mathbf{A}, \mathbf{B})) = \\ b(\mathbf{A}, \mathbf{B}, \mathbf{C}) + b(\mathbf{B}, \mathbf{C}, \mathbf{A}) + b(\mathbf{C}, \mathbf{A}, \mathbf{B}) - b(\mathbf{A}, \mathbf{C}, \mathbf{B}) - b(\mathbf{C}, \mathbf{B}, \mathbf{A}) - b(\mathbf{B}, \mathbf{A}, \mathbf{C})$$

a.k.a. **Akivis Identity** or **Generalized Jacobi Identity**.

For any (nonassociative) algebra one obtains an Akivis algebra by identifying a with the **commutator** and b with the **associator**. The equation is then an identity if one resolves the double commutators and the associators.

Akivis Algebras were introduced in 1976 by M. A. Akivis as local algebras of **three-webs**.

Theorem
A local algebra (**tangent algebra**) of a differentiable **quasigroup** is an Akivis algebra. (This theorem is a generalization of **Lie's first theorem** for differentiable quasigroups).

Theorem (Shestakov)
Any Akivis algebra can be isomorphically embedded in the algebra of commutators and associators of a certain nonassociative algebra. (This generalizes the corresponding theorem for Lie algebras which says that every Lie algebra is isomorphic to a subalgebra of commutators of a certain associative algebra).
However an Akivis algebra does not in general uniquely determine a differentiable quasigroup. (See next theorem).

Theorem
Local Akivis algebras associated with **Moufang-** and **Bol-quasigroups** determine these quasigroups in a unique way. Monoassociative quasigroups however are only determined uniquely by prolonged Akivis algebras (which are **hyperalgebras**).

Akivis algebras are so called binary-ternary algebras and are a generalization of Lie algebras which are binary algebras only. Akivis algebras can be regarded as **prolonged Lie algebras (hyperalgebras)** and can describe tangent algebras of quasigroups up to third order. In case that higher orders are relevant, a generalization is required, which are called prolonged Akivis algebras.
Every Lie algebra, Akivis algebra or prolonged Akivis algebra is in fact a special case of a **Sabinin algebra**.
Akivis algebras therefore allow for a straightforward and natural generalization of Lie theory.

The relation between Akivis algebras and **Sabinin algebras** was clarified by Shestakov and Umirbaev (2002).They showed that free nonassociative algebras are the universal enveloping algebras of free Akivis algebras, just as free associative algebras are the universal enveloping algebras of free Lie algebras.

Examples

i) **Lie algebras**: As these are associative the right hand side is zero and the expression reduces to the **Jacobi identity**.
ii) **Alternative Algebras** (**Octonions**): As the associator is antisymmtrical, the right hand side collapses to 6 times the associator of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

Papers:

- Every Akivis Algebra is Linear ? - I. P. Shestakov local pct. 15

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Akivis Element

An **Akivis Element** in the **free nonassociative algebra** is a polynomial which can be expressed exclusively in terms of the **commutator** and the **associator** (e.g. no "standalone" product is allowed).

Every Akivis element is a **primitive element**, however the converse is not true.

Examples

In degree 4 there are six Akivis elements, two involving commutators only,

$$\begin{aligned}\mathfrak{A}_1(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [[[\mathbf{A},\mathbf{B}],\mathbf{C}],\mathbf{D}]\\ \mathfrak{A}_3(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [[\mathbf{A},\mathbf{B}],[\mathbf{C},\mathbf{D}]]\end{aligned}$$

and the others being a combination of a commutator and an associator,

$$\begin{aligned}\mathfrak{A}_2(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [[\mathbf{A},\mathbf{B},\mathbf{C}],\mathbf{D}]\\ \mathfrak{A}_5(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [\mathbf{A},[\mathbf{B},\mathbf{C}],\mathbf{D}]\\ \mathfrak{A}_4(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [[\mathbf{A},\mathbf{B}],\mathbf{C},\mathbf{D}]\\ \mathfrak{A}_6(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [[\mathbf{A},\mathbf{B},[\mathbf{C},\mathbf{D}]]\end{aligned}$$

(We stick to the numbering in [1]).

For $\mathfrak{A}_2$ the order of the nesting of the commutator and the associator is converse to the one of $\mathfrak{A}_4$, $\mathfrak{A}_5$ and $\mathfrak{A}_6$.

In degree 4 there are two primitive elements which are not Akivis elements, given by the **quaternators** $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$.
Contrary to the Akivis elements the quaternators cannot be built from commutators and associators only.

Every primitive multilinear nonassociative polynomial of degree 4 is a linear combination of permutations of these six Akivis elements and the two non-Akivis elements.

Resolved form

Using results found under **commutators of degree 4**, we can write

$$\begin{aligned}\mathfrak{A}_1(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= ((\mathbf{AB})\mathbf{C})\mathbf{D}-((\mathbf{BA})\mathbf{C})\mathbf{D}-(\mathbf{C}(\mathbf{AB}))\mathbf{D}+(\mathbf{C}(\mathbf{BA}))\mathbf{D}-\mathbf{D}((\mathbf{AB})\mathbf{C})+\mathbf{D}((\mathbf{BA})\mathbf{C})+\mathbf{D}(\mathbf{C}(\mathbf{AB}))-\mathbf{D}(\mathbf{C}(\mathbf{BA}))\\ \mathfrak{A}_3(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= (\mathbf{AB})(\mathbf{CD})-(\mathbf{AB})(\mathbf{DC})-(\mathbf{BA})(\mathbf{CD})+(\mathbf{BA})(\mathbf{DC})-(\mathbf{CD})(\mathbf{AB})+(\mathbf{DC})(\mathbf{AB})+(\mathbf{CD})(\mathbf{BA})-(\mathbf{DC})(\mathbf{BA})\end{aligned}$$

$\mathfrak{A}_1$ and $\mathfrak{A}_3$ contain all **association types** possible in degree 4.
Moreover

$$\begin{aligned}\mathfrak{A}_2(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= [[\mathbf{A},\mathbf{B},\mathbf{C}],\mathbf{D}]=((\mathbf{AB})\mathbf{C})\mathbf{D}-(\mathbf{A}(\mathbf{BC}))\mathbf{D}-\mathbf{D}((\mathbf{AB})\mathbf{C})+\mathbf{D}(\mathbf{A}(\mathbf{BC}))\\ \mathfrak{A}_5(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= [\mathbf{A},[\mathbf{B},\mathbf{C}],\mathbf{D}]=(\mathbf{A}(\mathbf{BC}))\mathbf{D}-\mathbf{A}((\mathbf{BC})\mathbf{D})-(\mathbf{A}(\mathbf{CB}))\mathbf{D}+\mathbf{A}((\mathbf{CB})\mathbf{D})\\ \mathfrak{A}_4(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= [[\mathbf{A},\mathbf{B}],\mathbf{C},\mathbf{D}]=((\mathbf{AB})\mathbf{C})\mathbf{D}-(\mathbf{AB})(\mathbf{CD})-((\mathbf{BA})\mathbf{C})\mathbf{D}+(\mathbf{BA})(\mathbf{CD})\\ \mathfrak{A}_6(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= [[\mathbf{A},\mathbf{B},[\mathbf{C},\mathbf{D}]]=(\mathbf{AB})(\mathbf{CD})-\mathbf{A}(\mathbf{B}(\mathbf{CD}))-(\mathbf{AB})(\mathbf{DC})+\mathbf{A}(\mathbf{B}(\mathbf{DC}))\end{aligned}$$

Once again all association types are covered. One finds the interesting chain $2\to 4, 4\to 1, 1\to 3, 3\to 5$.

Tensorial representation

$$\begin{aligned}(\mathfrak{A}_1)^{\tau}_{\mu\nu\rho\sigma} &= T^{\kappa}_{\mu\nu}T^{\lambda}_{\kappa\rho}T^{\tau}_{\lambda\sigma}\\ (\mathfrak{A}_3)^{\tau}_{\mu\nu\rho\sigma} &= T^{\kappa}_{\mu\nu}T^{\lambda}_{\rho\sigma}T^{\tau}_{\kappa\lambda}\\ (\mathfrak{A}_2)^{\tau}_{\mu\nu\rho\sigma} &= A^{\kappa}_{\mu\nu\rho}T^{\tau}_{\kappa\sigma}\\ (\mathfrak{A}_5)^{\tau}_{\mu\nu\rho\sigma} &= A^{\tau}_{\mu\kappa\sigma}T^{\kappa}_{\nu\rho}\\ (\mathfrak{A}_4)^{\tau}_{\mu\nu\rho\sigma} &= T^{\kappa}_{\mu\nu}A^{\tau}_{\kappa\rho\sigma}\\ (\mathfrak{A}_6)^{\tau}_{\mu\nu\rho\sigma} &= A^{\tau}_{\mu\nu\kappa}T^{\kappa}_{\rho\sigma}\end{aligned}$$

where $T^{\rho}_{\mu\nu}$ is the **torsion tensor** and $A^{\sigma}_{\mu\nu\rho}$ the **nonassociativity tensor**. (See also **nested commutators and associators**).

Papers:

- On Hopf Algebra Structures over Operads (2004) - R. Holtkamp local pct. 35
- [1] Polynomial Identities for Tangent Algebras of Monoassociative Loops (2011) - M. R. Bremner, S. Madrigada local pct. 3 prl. 10
- Spontaneous Compactification and Nonassociativity (2009) - E. K. Loginov local pct. 2

Theses:

- Free Akivis Algebra (2005) - Ş. Findik local (Turkish)

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Albert Algebra

The **Albert Algebra** $\mathfrak{H}_3(\tilde{\mathbb{O}})$ is an exceptional **Jordan algebra** of 3×3-matrices over the **octonions** (or **split octonions**). An element of the algebra is given by:

$$\mathbf{J}=\begin{pmatrix}\alpha & \mathbf{A} & \mathbf{B}\\ \mathbf{A}^* & \beta & \mathbf{C}\\ \mathbf{B}^* & \mathbf{C}^* & \gamma\end{pmatrix}$$

where α, β, γ are scalars and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ (split) octionions. It is **hermitesch**, as evidently $\mathbf{J}=(\mathbf{J}^T)^*$.

The product of two elements $\mathbf{J}_1$ and $\mathbf{J}_2$ of the algebra is given by the Jordan product

$$\mathbf{J}_1 \circ \mathbf{J}_2 = \frac{1}{2}\left(\mathbf{J}_1\mathbf{J}_2 + \mathbf{J}_2\mathbf{J}_1\right)$$

and is therefore commutative, however not associative, due to the non-associativity of the octonions.

The **automorphism group** of the Albert algebra is $\mathbf{F}_4$.

Papers:

- Constructions for Octonion and Exceptional Jordan Algebras (2000) - L. J. Rylands, D. E. Taylor local pct. 13 - Contains a multiplication table -

Links:

- WIKIPEDIA - Albert Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Algebra

> By weakening or deleting various postulates of common algebra, or by replacing one of more of the postulates by others, which are consistent with the remaining postulates, an enormous variety of systems can be studied. As some of these systems we have groupoids, quasigroups, loops, semi-groups, monoids, groups, rings, integral domains, lattices, division rings, Boolean rings, Boolean algebras, ...fields, vector spaces, Jordan algebras, and Lie algebras, the last two being examples of non-associative algebras.
> - Howard Eves -

An **Algebra** $\mathcal{A}$ over a **field** $\mathcal{F}$ is a **ring** $\mathcal{R}$, supplemented with a bilinear multiplication

$$\alpha(\mathbf{A}\cdot\mathbf{B}) = (\alpha\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(\alpha\mathbf{B}),\ \forall\alpha\in\mathcal{F},\mathbf{A},\mathbf{B}\in\mathcal{R}$$

making it a vector space over $\mathcal{F}$.

Furthermore, if the associative law

$$(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}),\ \forall\mathbf{A},\mathbf{B},\mathbf{C}\in\mathcal{A}$$

holds, we have an associative algebra.

Papers:

- Abraham Adrian Albert 1905-1972 (1974) - N. Jacobson local pct. 13
- Encyclopedia of Types of Algebras (2007) - J.-L. Loday local pct. 6
- Phase Spaces of Algebras (2010) - B. A. Kupershmidt local pct. 1

Videos:

- Abstract Algebra Lecture - B. Gross

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Algebraic Structure

An **Algebraic Structure** is a **mathematical structure** having some algebraic operations like multiplication or addition.

In full generality, an algebraic structure may use any number of sets and any number of axioms in its definition. The most commonly studied structures, however, usually involve only one or two sets and one or two binary operations.

Links:

- WIKIPEDIA - Outline of Algebraic Structures

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Algebras and Signature

Clifford algebras

In general, the Clifford algebra $Cl(p,q)$ is different from $Cl(q,p)$. Their even subalgebras are **isomorphic** $Cl^+(q,p) \cong Cl^+(p,q)$ and so are the *spin groups*, but the **pin groups** are not. Hence a different signature corresponds to a different **(s)pinor**.

See also:

- **Wick rotation**

Papers:

- Signature Change and Clifford Algebras (2001) - D. Miralles, J. M. Parra, J. Vaz, Jr. local pct. 9
- Should Metric Signature Matter in Clifford Algebra Formulations of Physical Theories? (1997) - W. M. Pezzaglia Jr., J. J. Adams local pct. 8

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Alternative Algebra

An **Alternative (nearly associative) Algebra** satisfies the two conditions:

Left (mono-)alternativity

$$[\mathbf{A},\mathbf{A},\mathbf{B}]=0 \Leftrightarrow (\mathbf{AA})\mathbf{B}=\mathbf{A}(\mathbf{AB})$$

Right (mono-)alternativity

$$[\mathbf{A},\mathbf{B},\mathbf{B}]=0 \Leftrightarrow (\mathbf{AB})\mathbf{B}=\mathbf{A}(\mathbf{BB})$$

The linearized form of these two identities is

$$[\mathbf{A},\mathbf{B},\mathbf{C}]=-[\mathbf{B},\mathbf{A},\mathbf{C}]$$

$$[\mathbf{A},\mathbf{B},\mathbf{C}]=-[\mathbf{A},\mathbf{C},\mathbf{B}]$$

By means of them one can also define a **Left-** and **Right-Alternator** $\mathbf{Alt}_l$ and $\mathbf{Alt}_R$ given by:

$$\begin{aligned}\mathbf{Alt}_l(\mathbf{A},\mathbf{B},\mathbf{C}) &\equiv [\mathbf{A},\mathbf{B},\mathbf{C}]+[\mathbf{B},\mathbf{A},\mathbf{C}]=(\mathbf{AB})\mathbf{C}-\mathbf{A}(\mathbf{BC})+(\mathbf{BA})\mathbf{C}-\mathbf{B}(\mathbf{AC})\\ \mathbf{Alt}_r(\mathbf{A},\mathbf{B},\mathbf{C}) &\equiv [\mathbf{A},\mathbf{B},\mathbf{C}]+[\mathbf{A},\mathbf{C},\mathbf{B}]=(\mathbf{AB})\mathbf{C}-\mathbf{A}(\mathbf{BC})+(\mathbf{AC})\mathbf{B}-\mathbf{A}(\mathbf{CB})\end{aligned}$$

which vanishes for left- and right-alternative algebras respectively.

Every alternative algebra is also **flexible** and **power-associative**.

Alternativity and flexibility together lead to

$$[\mathbf{A}_{\sigma(1)},\mathbf{A}_{\sigma(2)},\mathbf{A}_{\sigma(3)}]=\mathrm{sgn}(\sigma)[\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3]$$

i.e. the **associator** is totally antisymmetric in respect to the exchanges of any 2 variables.

This implies:

$$[\mathbf{A}_{(1},\mathbf{A}_2,\mathbf{A}_{3)}]=0$$

Furthermore one has the generalization of the **Jacobi identity**:

$$\mathbf{J}_c(\mathbf{A},\mathbf{B},\mathbf{C})=6[\mathbf{A},\mathbf{B},\mathbf{C}]$$

Any simple nonassociative alternative algebra is isomorphic to the **octonions**.

Left-power-alternativity

An algebra is called **left power-alternative** if $\forall\mathbf{A},\mathbf{B}\in\mathcal{A}$

$$\mathbf{A}^m(\mathbf{A}^n\mathbf{B})=\mathbf{A}^{m+n}\mathbf{B}$$

for any $m,n\in\mathbb{Z}$.
Setting $m=-1$ and $n=1$ one obtains the *strong left inverse property*.

Right-power-alternativity

An algebra is called **right power-alternative** if $\forall\mathbf{A},\mathbf{B}\in\mathcal{A}$

$$(\mathbf{B}\mathbf{A}^m)\mathbf{A}^n=\mathbf{B}\mathbf{A}^{m+n}$$

for any $m,n\in\mathbb{Z}$.
Setting $m=-1$ and $n=1$ one obtains the *strong right inverse property*.

An algebra is called **power-alternative** if it is both a right and left power-alternative.

Papers:

- Identities for the Associator in Alternative Algebras - M. Bremner, I. Hentzel pct. 10
- The Nucleus of the Free Alternative Algebra - I. R. Hentzel, L. A. Peresi pct. 1
- [1] Irreducible binary (?1, 1)-bimodules over Simple Finite-Dimensional Algebras - S. V. Pchelintsev Russian local pct. 0

Links:

- Kevin McCrimmon's Pre-Book on Alternative Algebras

Google books

- Mutations of Alternative Algebras - A. Elduque, H. C. Myung local bct. 15 brl. 9

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Amitsur-Levitzki Theorem

Papers:

- Minimal Identities for Algebras (1950) - A. S. Amitsur, J. Levitzki local pct. 259

Links

- WIKIPEDIA - Amitsur-Levitzki Theorem

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Anti-Commutator

The **Anti-Commutator** of two elements $\mathbf{A}, \mathbf{B}$ of an algebra $\mathcal{A}$ is defined as:

$$\{\mathbf{A}, \mathbf{B}\} = \mathbf{AB} + \mathbf{BA}$$

The anti-commutator can also be expressed in terms of the **Jordan product**:

$$\{\mathbf{A}, \mathbf{B}\} = 2\mathbf{A} \circ \mathbf{B}$$

An algebra that is obtained from an algebra $\mathcal{A}$ by replacing the product $\mathbf{AB}$ with the anti-commutator $\{\mathbf{A}, \mathbf{B}\}$ is denoted $\mathcal{A}^+$.
$\mathcal{A}^+$ is called a **Commutative Algebra**.

An algebra $\mathcal{A}$ is called **Jordan-admissible** if the algebra $\mathcal{A}^+$ is a **Jordan algebra**.

See also:

- **Commutator**
- *Anticommutative algebra*

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Association Type Identities

An **Association Type Identity** will be understood as an equality of two **association types** W_1 and W_2, having the same degree n and rank r. It will be designated as a (n, r)-identity.
I.e.

$$\mathbf{W}_1(\mathbf{A}_1, \ldots, \mathbf{A}_n) = \mathbf{W}_2(\mathbf{A}_1, \ldots, \mathbf{A}_n)$$

We furthermore require:

- The identity is **Balanced**, which means that each of the variables enters the words W_1 and W_2 with the same multiplicity. (In smooth loops only these identities make sense).
- The identity does not have inversions, i.e. the order of the variables in W_1 and W_2 is the same.
- The identity is **Irreducible**, i.e. it cannot equivalently be expressed as an identity of smaller length. (Example: $(\mathbf{AB})(\mathbf{CD}) = \mathbf{A}(\mathbf{B}(\mathbf{CD}))$ can be reduced to $(\mathbf{AB})\mathbf{C}' = \mathbf{A}(\mathbf{BC}')$).

A balanced identity without inversions is also called **Regular**.
Concerning the classical **3-webs** R, M, B and H, all nontrivial webs are characterized by regular identities.

For association types of degree n and rank r one has the following identities:

n=2

r=1

Trivial

r=2

$\mathbf{AB} = \mathbf{BA} \Leftrightarrow [\mathbf{A}, \mathbf{B}] = 0$, which is condition of **commutativity**. (Actually this identity should be excluded from this list, as it is an inversion). It applies to **Abelian groups**.

n=3

r=1

$(\mathbf{AA})\mathbf{A} = \mathbf{A}(\mathbf{AA}) \Leftrightarrow [\mathbf{A}, \mathbf{A}, \mathbf{A}] = 0$, which is the **monoassociativity condition**.

r=2

$(\mathbf{AA})\mathbf{B} = \mathbf{A}(\mathbf{AB}) \Leftrightarrow [\mathbf{A}, \mathbf{A}, \mathbf{B}] = 0$, which is the **left alternativity law**.
$(\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA}) \Leftrightarrow [\mathbf{A}, \mathbf{B}, \mathbf{A}] = 0$, which is the **flexibility law**.
$(\mathbf{BA})\mathbf{A} = \mathbf{B}(\mathbf{AA}) \Leftrightarrow [\mathbf{B}, \mathbf{A}, \mathbf{A}] = 0$, which is the **right alternativity law**.

r=3

$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \Leftrightarrow [\mathbf{A}, \mathbf{B}, \mathbf{C}] = 0$, which is the **associativity condition** that applies to Abelian as well as to non-Abelian groups.

n=4

As the number of identities proliferates in this case one can easily get lost. We therefore introduce a more compact notation, making it easier to compare the different identities.
It has the form:

$$A_1 A_2 V_1 V_2 V_3 V_4$$

where A_1 and A_2 stand for the **numbers** of the two association types occurring and $V_1, \ldots, V_4$ are either "X" in case of the variable that occurs repeatedly or "0" else.
Example:
$(\mathbf{A}(\mathbf{BA}))\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{AC}))$ translates into $45X0X0$.
Furthermore we'll (try to) stick to the convention to put the association type with the smaller number to the l.h.s. of the equation.
If it is clear from the context, the string $A_1 A_2 V_1 V_2 V_3 V_4$ may also be appropriately curtailed. E.g. if it is always $XXXX$, it can even be left away.

r=1

Identity	Abbreviation
$((\mathbf{AA})\mathbf{A})\mathbf{A} = \mathbf{A}((\mathbf{AA})\mathbf{A})$	12
$((\mathbf{AA})\mathbf{A})\mathbf{A} = (\mathbf{AA})(\mathbf{AA}) \Leftrightarrow [\mathbf{A}^2, \mathbf{A}, \mathbf{A}] = 0$	13
$((\mathbf{AA})\mathbf{A})\mathbf{A} = (\mathbf{A}(\mathbf{AA}))\mathbf{A}$	14
$((\mathbf{AA})\mathbf{A})\mathbf{A} = \mathbf{A}(\mathbf{A}(\mathbf{AA}))$	15
$\mathbf{A}((\mathbf{AA})\mathbf{A}) = (\mathbf{AA})(\mathbf{AA})$	23
$\mathbf{A}((\mathbf{AA})\mathbf{A}) = (\mathbf{A}(\mathbf{AA}))\mathbf{A} \Leftrightarrow [\mathbf{A}, \mathbf{A}^2, \mathbf{A}] = 0$	24
$\mathbf{A}((\mathbf{AA})\mathbf{A}) = \mathbf{A}(\mathbf{A}(\mathbf{AA}))$	25
$(\mathbf{AA})(\mathbf{AA}) = (\mathbf{A}(\mathbf{AA}))\mathbf{A}$	34
$(\mathbf{AA})(\mathbf{AA}) = \mathbf{A}(\mathbf{A}(\mathbf{AA})) \Leftrightarrow [\mathbf{A}, \mathbf{A}, \mathbf{A}^2] = 0$	35
$(\mathbf{A}(\mathbf{AA}))\mathbf{A} = \mathbf{A}(\mathbf{A}(\mathbf{AA}))$	45

If one assumes monoassociativity, the identities 14 and 25 are trivial.

r=2

See **(4,2) - association type identities**.

r=3

See **(4,3) - association type identities**.

r=4

One has the same identities as in the case $r = 1$, except for the fact that all elements are different. So in a sense, this case can regarded as "dual" to the rank 1-case.

Identity	Abbreviation	Reducibility
$((\mathbf{AB})\mathbf{C})\mathbf{D} = \mathbf{A}((\mathbf{BC})\mathbf{D})$	12	
$((\mathbf{AB})\mathbf{C})\mathbf{D} = (\mathbf{AB})(\mathbf{CD}) \Leftrightarrow [\mathbf{AB}, \mathbf{C}, \mathbf{D}] = 0$	13	Associative
$((\mathbf{AB})\mathbf{C})\mathbf{D} = (\mathbf{A}(\mathbf{BC}))\mathbf{D} \Leftrightarrow [\mathbf{A}, \mathbf{B}, \mathbf{C}]\mathbf{D} = 0$	14	Associative
$((\mathbf{AB})\mathbf{C})\mathbf{D} = \mathbf{A}(\mathbf{B}(\mathbf{CD}))$	15	
$\mathbf{A}((\mathbf{BC})\mathbf{D}) = (\mathbf{AB})(\mathbf{CD})$	23	
$\mathbf{A}((\mathbf{BC})\mathbf{D}) = (\mathbf{A}(\mathbf{BC}))\mathbf{D} \Leftrightarrow [\mathbf{A}, \mathbf{BC}, \mathbf{D}] = 0$	24	Associative
$\mathbf{A}((\mathbf{BC})\mathbf{D}) = \mathbf{A}(\mathbf{B}(\mathbf{CD})) \Leftrightarrow \mathbf{A}[\mathbf{B}, \mathbf{C}, \mathbf{D}] = 0$	25	Associative
$(\mathbf{AB})(\mathbf{CD}) = (\mathbf{A}(\mathbf{BD}))\mathbf{A}$	34	
$(\mathbf{AB})(\mathbf{CD}) = \mathbf{A}(\mathbf{B}(\mathbf{CD})) \Leftrightarrow [\mathbf{A}, \mathbf{B}, \mathbf{CD}] = 0$	35	Associative
$(\mathbf{A}(\mathbf{BC}))\mathbf{D} = \mathbf{A}(\mathbf{B}(\mathbf{CD}))$	45	

Consequently there are only 5 non-trivial identities of degree four.

Generally speaking, the lower the rank of an identity, the weaker is the constraint it imposes on an algebra. This is due to the fact that an identity of a given rank (except for one of maximal rank) is always contained in an identity of a higher rank (i.e. is a special case of it). The identity of the lower rank can be obtained by identifying variables in the identity of the higher rank appropriately.

Combining several association type identities leads to (what we'll call) *fundamental (quasigroup) identities*.

Papers:

- The Varieties of Quasigroups of Bol-Moufang Type: An Equational Reasoning Approach (2005) - J. D. Phillips, P. Vojtěchovský local pct. 26

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Banach Algebra

An algebra $\mathcal{A}$ is called a **Banach Algebra** if there is a *submultiplicative* **norm** on $\mathcal{A}$ such that

$$||\mathbf{AB}|| \le ||\mathbf{A}||\,||\mathbf{B}||$$

$\forall \mathbf{A}, \mathbf{B}$ and $\mathcal{A}$ is complete (i.e. any Cauchy sequence converges).

A **Banach *-Algebra** or **Involutive Banach Algebra** $\mathcal{A}$ (over $\mathbb{C}$) ist a Banach Algebra, supplemented with an **involution** *, satisfying:

- $(\mathbf{A}^*)^* = \mathbf{A}$
- $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$
- $(\lambda\mathbf{A} + \mu\mathbf{B})^* = \bar{\lambda}\mathbf{A}^* + \bar{\mu}\mathbf{A}^*, \quad \text{Semi-linearity, Anti-linearity or Conjugate Linearity}$
- $\|\mathbf{A}\| = \|\mathbf{A}^*\|, \quad \text{Isometry}$

$\forall \mathbf{A}, \mathbf{B} \in \mathcal{A}$, $\forall \lambda, \mu \in \mathbb{C}$, with $\bar{\mu}$ designating the complex conjugatie of μ.

Links:

- WIKIPEDIA - Banach Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Bialgebra

A **Bialgebra** over a **field** $\mathbb{K}$ is a vector space over $\mathbb{K}$ which is both a unital associative algebra and a *coalgebra*, such that the algebraic- and coalgebraic structure satisfy certain compatiblility relations.

Links:

- WIKIPEDIA - Bialgebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Bioctonion

The **Bioctonion** algebra $\mathbb{O}_\mathbb{C}$ is obtained by complexifying the (real) **octonion** algebra $\mathbb{O}$,

$$\mathbb{O}_\mathbb{C} \equiv \mathbb{O} \otimes \mathbb{C} = \{\mathbf{C} = \mathbf{A} + \mathbf{i}_\mathbb{C}\mathbf{B} \mid \mathbf{A}, \mathbf{B} \in \mathbb{O},\ \mathbf{i}_\mathbb{C}^2 = -1\}$$

where the (complex) imaginary unit $\mathbf{i}_\mathbb{C}$ is assumed to commute with all imaginary basis units $\mathbf{e}_j$ ($j = 1, 2, \ldots, 7$) of $\mathbb{O}$. The algebra therefore also goes under the name **Complex Octonions**.

Setting $\mathbf{A} = a_i\mathbf{e}_i$ and $\mathbf{B} = b_i\mathbf{e}_i$ one gets

$$\mathbf{C} = \sum_{i=0}^{7}(a_i + \mathbf{i}_\mathbb{C} b_i)\mathbf{e}_i$$

Thus bioctonions are octonions having complex coefficients ($a_i + \mathbf{i}_\mathbb{C} b_i$ in this case).

Contrary to the *bicomplex numbers* which are **isomorphic** to $\mathbb{C} \oplus \mathbb{C}$, and the **biquaternions** which are isomorphic to 2×2 complex matrices, the bioctonions are really "something new".

Multiplication Table

	$\mathbb{O}$	$\mathbf{i}_\mathbb{C}\ \mathbb{O}$
$\mathbb{O}$	$\mathbb{O} \cdot \mathbb{O}$	$\mathbf{i}_\mathbb{C}\ \mathbb{O} \cdot \mathbb{O}$
$\mathbf{i}_\mathbb{C}\ \mathbb{O}$	$\mathbf{i}_\mathbb{C}\ \mathbb{O} \cdot \mathbb{O}$	$-\mathbb{O} \cdot \mathbb{O}$

with $\mathbb{O} \cdot \mathbb{O}$ denoting the **multiplication table of the octonions**.
The multiplication table is symmetrical, having a "balanced" **signature**.

Conjugation

Different types of bioctonion conjugations are possible:

Complex conjugation

$$\begin{aligned}\bar{\mathbf{C}} &\equiv \mathbf{A} + \mathbf{i}_{\mathbb{C}}^{*}\mathbf{B} \\ &= \mathbf{A} - \mathbf{i}_{\mathbb{C}}\mathbf{B}\end{aligned}$$

Octonionic conjugation

$$\begin{aligned}\mathbf{C}^{*} &\equiv \mathbf{A}^{*} + \mathbf{i}_{\mathbb{C}}\mathbf{B}^{*} \\ &= (a_0 + \mathbf{i}_{\mathbb{C}} b_0)\mathbf{e} - \sum_{i=1}^{7}(a_i + \mathbf{i}_{\mathbb{C}} b_i)\mathbf{e}_i\end{aligned}$$

which is similar to the conjugation in the **Cayley-Dickson algebras**, where only the sign of the identity basis element is not flipped. This conjugation allows for the definition of a **quadratic form** $Q : \mathbb{O}_{\mathbb{C}} \to \mathbb{C}$ according to

$$\begin{aligned}\mathcal{Q}(\mathbf{C}) &\equiv \mathbf{C}\mathbf{C}^{*} \\ &= (\mathbf{A} + \mathbf{i}_{\mathbb{C}}\mathbf{B})(\mathbf{A}^{*} + \mathbf{i}_{\mathbb{C}}\mathbf{B}^{*}) \\ &= ||\mathbf{A}||^2 - ||\mathbf{B}||^2 + \mathbf{i}_{\mathbb{C}}(\mathbf{A}\mathbf{B}^{*} + \mathbf{B}\mathbf{A}^{*}) \\ &= ||\mathbf{A}||^2 - ||\mathbf{B}||^2 + \mathbf{i}_{\mathbb{C}}(\mathbf{A}\mathbf{B}^{*} + (\mathbf{A}\mathbf{B}^{*})^{*}) \\ &= ||\mathbf{A}||^2 - ||\mathbf{B}||^2 + 2\mathbf{i}_{\mathbb{C}}\,\mathrm{Re}(\mathbf{A}\mathbf{B}^{*}) \\ &= \sum_{i=0}^{7}(a_i^2 - b_i^2) + 2\mathbf{i}_{\mathbb{C}}\langle\mathbf{A}|\mathbf{B}\rangle_{\mathbb{R}}\end{aligned}$$

with real **inner product** $\langle\mathbf{A}|\mathbf{B}\rangle_{\mathbb{R}}$.

Properties

The bioctonion algebra ...

- inherits the algebraic properties of its octonionic cousin (e.g. it is noncommutative, **nonassociative**, **alternative**, **flexible** and **power-associative**),
- contains **zero divisors** (e.g. $(\mathbf{e}_1 + i\mathbf{e}_0)(\mathbf{e}_1 - i\mathbf{e}_0)$),
- is isomorphic to the **Conic Sedenions** of **Musès' hypercomplex number system**,
- can be made a **composition algebra** over $\mathbb{C}$. By application of the **Moufang identities** for the octonions, it can be shown that $\mathcal{Q}(\mathbf{C}_1\mathbf{C}_2) = \mathcal{Q}(\mathbf{C}_1)\mathcal{Q}(\mathbf{C}_2)\ \forall\, \mathbf{C}_1, \mathbf{C}_2 \in \mathbb{O}_{\mathbb{C}}$,
- contains both, a real octonion algebra $\mathbb{O}$ and a **split octonion** algebra $\mathbb{O}_S$ as subalgebra,
- is a *noncommutative Jordan C*-Algebra*,
- has **G2(C)** as its **automorphism group**.

Some proofs

We show that the bioctonions are a **noncommutative Jordan algebra** and hence are power associative.

Flexibility

Let $\mathbf{A}, \mathbf{B} \in \mathbb{O}_{\mathbb{C}}$ with $\mathbf{A} \equiv \mathbf{A}_1 + \mathbf{i}_{\mathbb{C}}\mathbf{A}_2$ and $\mathbf{B} \equiv \mathbf{B}_1 + \mathbf{i}_{\mathbb{C}}\mathbf{B}_2$, then

$$\begin{aligned}[\mathbf{A},\mathbf{B},\mathbf{A}] = \ & [\mathbf{A}_1,\mathbf{B}_1,\mathbf{A}_1] + [\mathbf{A}_1,\mathbf{B}_1,\mathbf{A}_2]\mathbf{i}_{\mathbb{C}} + [\mathbf{A}_2,\mathbf{B}_1,\mathbf{A}_1]\mathbf{i}_{\mathbb{C}} - [\mathbf{A}_2,\mathbf{B}_1,\mathbf{A}_2] + \\ & [\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_1]\mathbf{i}_{\mathbb{C}} - [\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_2] - [\mathbf{A}_2,\mathbf{B}_2,\mathbf{A}_1] - [\mathbf{A}_2,\mathbf{B}_2,\mathbf{A}_2]\mathbf{i}_{\mathbb{C}}\end{aligned}$$

Due to flexibility of the octonions, the first, fourth, fifth and last term drop out. Thus if one requires flexibility for the bioctonions, one is left with the condition

$$[\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_2] + [\mathbf{A}_2,\mathbf{B}_2,\mathbf{A}_1] + ([\mathbf{A}_1,\mathbf{B}_1,\mathbf{A}_2] + [\mathbf{A}_2,\mathbf{B}_1,\mathbf{A}_1])\mathbf{i}_{\mathbb{C}} \equiv \mathbf{0} + \mathbf{0}\mathbf{i}_{\mathbb{C}}$$

Yet this is nothing but two times the polarized form of the flexibility condition for the octonions.

Jordan identity

As flexibility holds for the bioctonions, we can pick either one of the **Jordan identities** which is satisfied for the octonions for showing that it is also satisfied for the bictonions. We choose $[\mathbf{A}, \mathbf{B}, \mathbf{A}^2] = 0$. Ananlogously to above we get the condition

$$[\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_2^2] + [\mathbf{A}_2,\mathbf{B}_2,\mathbf{A}_1^2] + ([\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_2^2] + [\mathbf{A}_2,\mathbf{B}_2,\mathbf{A}_1^2])\mathbf{i}_{\mathbb{C}} \equiv \mathbf{0} + \mathbf{0}\mathbf{i}_{\mathbb{C}}$$

which once again applying two times the polarized form of the octonions is satisfied.

See also:

- **Bioctonionic projective plane**
- **Quateroctonion**
- *Octooctonion*
- **Bisedenion**

Papers:

- Sedenionic Formulation for Generalized Fields of Dyons (2011) - S. Demir, M. Tanışlı local pct. 9
- Explicit Classification of Orbits in Cayley Algebras over the Groups of Type G_2 (2007) - O. Shukuzawa local pct. 9
- Unified Theory of Ideals (2010) - C. Furey local pct. 8
- $Sp(4,H)/Z_2$ Pair Universe in E_6 Matrix Models (2005) - Y. Ohwashi local pct. 3 - In particular the appendix. -
- Jordan C*-Algebras and Supergravity (2010) - M. Rios local pct. 2
- Grassmann Variables in Jordan Matrix Models (2005) - M. Rios local pct. 2
- Electroweak Interaction without Projection Operators using Complexified Octonions (2011) - J. Fredsted local pct. 0
- Natural Octonionic Generalization of General Relativity (2007) - J. Fredsted local pct. 0

Links:

- Physics Forums

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Biquaternion

> The independence of $\mathbf{i}$ and $\mathbf{i}_\mathbb{C}$ is perhaps the most fundamental axiomatic aspect of the biquaternions that must be understood.
> - [1] (notation adapted) -

The algebra of **Biquaternions** $\mathbb{H}_\mathbb{C}$ is obtained by complexifying the **quaternion algebra** $\mathbb{H}$, i.e. $\mathbb{H}_\mathbb{C} = \mathbb{C} \otimes \mathbb{H}$. They therefore also go under the name **Complex Quaternions**.

Contrary to the quaternion algebra, $\mathbb{H}_\mathbb{C}$ contains **idempotents** (elements whose square remains unchanged) and **nilpotents** (elements whose square vanishes). It also contains **divisors of zero** (elements with vanishing **norm**) and thus is not a **division algebra**. The idempotents and nilpotents are subsets of the divisors of zero.

The biquaternions are **isomorphic** to the algebra of 2×2 complex matrices.
In terms of **Clifford algebra** they can be classified as $Cl(2;\mathbb{C}) = Cl^0(3;\mathbb{C})$. This is also isomorphic to the **Pauli algebra** $Cl(3,0)$, and the even part of the **space-time algebra** $Cl^0(1,3)$.

As is the case for the quaternions, the biquaternion product is associative but not commutative.

The **automorphism group** of the biquaternions is *SO(3,C)* which is 6-parametric and isomorphic to the (proper) **Lorentz group**.

Conjugations

There exist three different conjugations over $\mathbb{H}_\mathbb{C}$. Thus, given a complex quaternion $\mathbf{A}$, it is possible to define its:

- **Complex** (or $\mathbb{C}$-) **Conjugate**: $\bar{\mathbf{A}} \equiv \mathbf{A}_0^* + \mathbf{A}_1^*\mathbf{i} + \mathbf{A}_2^*\mathbf{j} + \mathbf{A}_3^*\mathbf{k} = \Re_\mathbb{C}(\mathbf{A}) - \mathbf{i}_\mathbb{C}\Im_\mathbb{C}(\mathbf{A})$
- **Quaternionic** (or $\mathbb{H}$-) **Conjugate**: $\mathbf{A}^* \equiv \mathbf{A}_0 - \mathbf{A}_1\mathbf{i} - \mathbf{A}_2\mathbf{j} - \mathbf{A}_3\mathbf{k}$
- **Total/Bi-** (or $\mathbb{H}_\mathbb{C}$-) **Conjugate**: $\mathbf{A}^\star \equiv \bar{\mathbf{A}}^* = \mathbf{A}_0^* - \mathbf{A}_1^*\mathbf{i} - \mathbf{A}_2^*\mathbf{j} - \mathbf{A}_3^*\mathbf{k}$

These definitions induce different possible definitions for **norms** (see "inner products" below).

Properties

$$\begin{aligned} \overline{\mathbf{AB}} &= \bar{\mathbf{A}}\bar{\mathbf{B}} \\ (\mathbf{AB})^* &= \mathbf{B}^*\mathbf{A}^* \\ \overline{\mathbf{A}^*} &= \bar{\mathbf{A}}^* \end{aligned}$$

but

$$\mathbf{A}\bar{\mathbf{A}} \neq \bar{\mathbf{A}}\mathbf{A}$$

Consequently

$$(\mathbf{AB})^\star = \mathbf{B}^\star\mathbf{A}^\star$$

Inner Products

I.

I.

$$\langle \mathbf{A}|\mathbf{B}\rangle_\mathbb{C} \equiv \frac{1}{2}\,(\mathbf{B}^*\mathbf{A} + \mathbf{A}^*\mathbf{B}) = \frac{1}{2}\,(\mathbf{B}\mathbf{A}^* + \mathbf{A}\mathbf{B}^*)$$

This implies a **Pseudo-Norm** $||\,.\,||_\mathbb{C}$, given by:

$$||\mathbf{A}||_\mathbb{C}^2 = \langle \mathbf{A}|\mathbf{A}\rangle_\mathbb{C} = \mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^* = \mathbf{A}_0^2 + \mathbf{A}_1^2 + \mathbf{A}_2^2 + \mathbf{A}_3^2$$

and it satisfies the following equality: $||\mathbf{AB}||_\mathbb{C}^2 = ||\mathbf{A}||_\mathbb{C}^2\,||\mathbf{B}||_\mathbb{C}^2 \;\; \forall \mathbf{A},\mathbf{B} \in \mathbb{H}_\mathbb{C}$.

It has the drawback of being complex valued in general. This involves that the pseudo-norm of a non-zero biquaternion can vanish.

II.

...

Physical Applications

Biquaternions can be used to represent *Lorentz transformations*.

Historical

The complexified quaternions, or biquaternions, were discovered by Hamilton himself and published within ten years of the discovery of quaternions.

Multiplication Table

	$\mathbb{H}$	$\mathbf{i}_{\mathbb{C}}\ \mathbb{H}$
$\mathbb{H}$	$\mathbb{H}\cdot\mathbb{H}$	$\mathbf{i}_{\mathbb{C}}\ \mathbb{H}\cdot\mathbb{H}$
$\mathbf{i}_{\mathbb{C}}\ \mathbb{H}$	$\mathbf{i}_{\mathbb{C}}\ \mathbb{H}\cdot\mathbb{H}$	$-\mathbb{H}\cdot\mathbb{H}$

with $\mathbb{H}\cdot\mathbb{H}$ denoting the **multiplication table of the quaternions**.
The multiplication table is symmetrical, having a "balanced" **signature**. As for block matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$, $\det\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \mathbf{AD} - \mathbf{BC}$, the determinant of the multiplication table is zero.

See also:

- **Split-biquaternion**
- *Bicomplex number*
- **Bioctonion**

Papers:

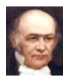
- MUSIC Algorithm for Vector-Sensors Array Using Biquaternions (2006) - N. Le Bihan, S. Miron, J. I. Mars local pct. 56
- Biquaternionic Proca-type generalization of Gravity (2011) - S. Demira and M. Tanışlı local pct. 14
- The Physical Heritage of Sir W.R. Hamilton (2009) - A. Gsponer, J.-P. Hurni local pct. 13
- [1] Fundamental Representations and Algebraic Properties of Biquaternions or Complexified Quaternions (2010) - S. J. Sangwine, T. A. Ell, N. Le Bihan local pct. 8
- Spinors in the Hyperbolic Algebra (2006) - S. Ulrych local pct. 7
- Determination of the Biquaternion Divisors of Zero, including the Idempotents and Nilpotents (2008) - S. J. Sangwine, D. Alfsmann local pct. 4
- COMPLEX QUATERNIONS AND SPINOR REPRESENTATIONS OF DE SITTER GROUPS SO(4,1) AND SO(3,2) (1978) - R. M. Mir-Kasïmov, I. P. Volobujev local pct. 3
- Algebrodynamics in Complex Space-time and the Complex-quaternionic Origin of Minkowski Geometry (2006) - V. V. Kassandrov local pct. 2

Documents:

- Bogus Reasons Can Give Right Covariance (2005) - J. D. Edmonds Jr. local

Links:

- WIKIPEDIA - Biquaternion



Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Bisedenion

The **Bisedenions** will be understood as the algebra of complexified **(Cayey-Dickson) sedenions**, i.e. $\mathbb{S}_{\mathbb{C}} \equiv \mathbb{C}\otimes\mathbb{S}$. We'll therefore alternatively refer to them as **Complex Sedenions**. (As sometimes the **complex octonions** are denoted conic sedenions, it would also be consequent to call them **Conic Tritintaduonions**. But as this designation is apt to lead to confusions we'll avoid it here).

The bisedenions inherit the algebraic properties of the sedenions. E.g. they satisfy the **Jordan identity**, are **flexible** and thus are a **power-associative algbebra**

The multiplication table, just for fun:

	ee1	ee2	ee3	ee4	ee5	ee6	ee7	ee8	ee9	ee10	ee11	ee12	ee13	ee14	ee15
ee1	-e1	ee3	-ee2	ee5	-ee4	-ee7	ee6	ee9	-ee8	-ee11	ee10	-ee13	ee12	ee15	-ee14
ee2	-ee3	-e1	ee1	ee6	ee7	-ee4	-ee5	ee10	ee11	-ee8	-ee9	-ee14	-ee15	ee12	ee13
ee3	ee2	-ee1	-e1	ee7	-ee6	ee5	-ee4	ee11	-ee10	ee9	-ee8	-ee15	ee14	-ee13	ee12
ee4	-ee5	-ee6	-ee7	-e1	ee1	ee2	ee3	ee12	ee13	ee14	ee15	-ee8	-ee9	-ee10	-ee11
ee5	ee4	-ee7	ee6	-ee1	-e1	-ee3	ee2	ee13	-ee12	ee15	-ee14	ee9	-ee8	ee11	-ee10

	i1	ie1	ie2	ie3	ie4	ie5	ie6	ie7	ie8	ie9	ie10	ie11	ie12	ie13	ie14	ie15
ee1	ie1	-i1	ie3	-ie2	ie5	-ie4	-ie7	ie6	ie9	-ie8	-ie11	ie10	-ie13	ie12	ie15	-ie14
ee2	ie2	-ie3	-i1	ie1	ie6	ie7	-ie4	-ie5	ie10	ie11	-ie8	-ie9	-ie14	-ie15	ie12	ie13
ee3	ie3	ie2	-ie1	-i1	ie7	-ie6	ie5	-ie4	ie11	-ie10	ie9	-ie8	-ie15	ie14	-ie13	ie12
ee4	ie4	-ie5	-ie6	-ie7	-i1	ie1	ie2	ie3	ie12	ie13	ie14	ie15	-ie8	-ie9	-ie10	-ie11
ee5	ie5	ie4	-ie7	ie6	-ie1	-i1	-ie3	ie2	ie13	-ie12	ie15	-ie14	ie9	-ie8	ie11	-ie10

ee6	ee7	ee4	-ee5	-ee2	ee3	-e1	-ee1	ee14	-ee15	-ee12	ee13	ee10	-ee11	-ee8	ee9	ie6	ie7	ie4	-ie5	-ie2	ie3	-i1	-ie1	ie14	-ie15	-ie12	ie13	ie10	-ie11	-ie8	ie9
ee7	-ee6	ee5	ee4	-ee3	-ee2	ee1	-e1	ee15	ee14	-ee13	-ee12	ee11	ee10	-ee9	-ee8	ie7	-ie6	ie5	ie4	-ie3	-ie2	ie1	-i1	ie15	ie14	-ie13	-ie12	ie11	ie10	-ie9	-ie8
ee8	-ee9	-ee10	-ee11	-ee12	-ee13	-ee14	-ee15	-e1	ee1	ee2	ee3	ee4	ee5	ee6	ee7	ie8	-ie9	-ie10	-ie11	-ie12	-ie13	-ie14	-ie15	-i1	ie1	ie2	ie3	ie4	ie5	ie6	ie7
ee9	ee8	-ee11	ee10	-ee13	ee12	ee15	-ee14	-ee1	-e1	-ee3	ee2	-ee5	ee4	ee7	-ee6	ie9	ie8	-ie11	ie10	-ie13	ie12	ie15	-ie14	-ie1	-i1	-ie3	ie2	-ie5	ie4	ie7	-ie6
ee10	ee11	ee8	-ee9	-ee14	-ee15	ee12	ee13	-ee2	ee3	-e1	-ee1	-ee6	-ee7	ee4	ee5	ie10	ie11	ie8	-ie9	-ie14	-ie15	ie12	ie13	-ie2	ie3	-i1	-ie1	-ie6	-ie7	ie4	ie5
ee11	-ee10	ee9	ee8	-ee15	ee14	-ee13	ee12	-ee3	-ee2	ee1	-e1	-ee7	ee6	-ee5	ee4	ie11	-ie10	ie9	ie8	-ie15	ie14	-ie13	ie12	-ie3	-ie2	ie1	-i1	-ie7	ie6	-ie5	ie4
ee12	ee13	ee14	ee15	ee8	-ee9	-ee10	-ee11	-ee4	ee5	ee6	ee7	-e1	-ee1	-ee2	-ee3	ie12	ie13	ie14	ie15	ie8	-ie9	-ie10	-ie11	-ie4	ie5	ie6	ie7	-i1	-ie1	-ie2	-ie3
ee13	-ee12	ee15	-ee14	ee9	ee8	ee11	-ee10	-ee5	-ee4	ee7	-ee6	ee1	-e1	ee3	-ee2	ie13	-ie12	ie15	-ie14	ie9	ie8	ie11	-ie10	-ie5	-ie4	ie7	-ie6	ie1	-i1	ie3	-ie2
ee14	-ee15	-ee12	ee13	ee10	-ee11	ee8	ee9	-ee6	-ee7	-ee4	ee5	ee2	-ee3	-e1	ee1	ie14	-ie15	-ie12	ie13	ie10	-ie11	ie8	ie9	-ie6	-ie7	-ie4	ie5	ie2	-ie3	-i1	ie1
ee15	ee14	-ee13	-ee12	ee11	ee10	-ee9	ee8	-ee7	ee6	-ee5	-ee4	ee3	ee2	-ee1	-e1	ie15	ie14	-ie13	-ie12	ie11	ie10	-ie9	ie8	-ie7	ie6	-ie5	-ie4	ie3	ie2	-ie1	-i1
i1	ie1	ie2	ie3	ie4	ie5	ie6	ie7	ie8	ie9	ie10	ie11	ie12	ie13	ie14	ie15	-e1	-ee1	-ee2	-ee3	-ee4	-ee5	-ee6	-ee7	-ee8	-ee9	-ee10	-ee11	-ee12	-ee13	-ee14	-ee15
ie1	-i1	ie3	-ie2	ie5	-ie4	-ie7	ie6	ie9	-ie8	-ie11	ie10	-ie13	ie12	ie15	-ie14	-ee1	e1	-ee3	ee2	-ee5	ee4	ee7	-ee6	-ee9	ee8	ee11	-ee10	ee13	-ee12	-ee15	ee14
ie2	-ie3	-i1	ie1	ie6	ie7	-ie4	-ie5	ie10	ie11	-ie8	-ie9	-ie14	-ie15	ie12	ie13	-ee2	ee3	e1	-ee1	-ee6	-ee7	ee4	ee5	-ee10	-ee11	ee8	ee9	ee14	ee15	-ee12	-ee13
ie3	ie2	-ie1	-i1	ie7	-ie6	ie5	-ie4	ie11	-ie10	ie9	-ie8	-ie15	ie14	-ie13	ie12	-ee3	-ee2	ee1	e1	-ee7	ee6	-ee5	ee4	-ee11	ee10	-ee9	ee8	ee15	-ee14	ee13	-ee12
ie4	-ie5	-ie6	-ie7	-i1	ie1	ie2	ie3	ie12	ie13	ie14	ie15	-ie8	-ie9	-ie10	-ie11	-ee4	ee5	ee6	ee7	e1	-ee1	-ee2	-ee3	-ee12	-ee13	-ee14	-ee15	ee8	ee9	ee10	ee11
ie5	ie4	-ie7	ie6	-ie1	-i1	-ie3	ie2	ie13	-ie12	ie15	-ie14	ie9	-ie8	ie11	-ie10	-ee5	-ee4	ee7	-ee6	ee1	e1	ee3	-ee2	-ee13	ee12	-ee15	ee14	-ee9	ee8	-ee11	ee10
ie6	ie7	ie4	-ie5	-ie2	ie3	-i1	-ie1	ie14	-ie15	-ie12	ie13	ie10	-ie11	-ie8	ie9	-ee6	-ee7	-ee4	ee5	ee2	-ee3	e1	ee1	-ee14	ee15	ee12	-ee13	-ee10	ee11	ee8	-ee9
ie7	-ie6	ie5	ie4	-ie3	-ie2	ie1	-i1	ie15	ie14	-ie13	-ie12	ie11	ie10	-ie9	-ie8	-ee7	ee6	-ee5	-ee4	ee3	ee2	-ee1	e1	-ee15	-ee14	ee13	ee12	-ee11	-ee10	ee9	ee8
ie8	-ie9	-ie10	-ie11	-ie12	-ie13	-ie14	-ie15	-i1	ie1	ie2	ie3	ie4	ie5	ie6	ie7	-ee8	ee9	ee10	ee11	ee12	ee13	ee14	ee15	e1	-ee1	-ee2	-ee3	-ee4	-ee5	-ee6	-ee7
ie9	ie8	-ie11	ie10	-ie13	ie12	ie15	-ie14	-ie1	-i1	-ie3	ie2	-ie5	ie4	ie7	-ie6	-ee9	-ee8	ee11	-ee10	ee13	-ee12	-ee15	ee14	ee1	e1	ee3	-ee2	ee5	-ee4	-ee7	ee6
ie10	ie11	ie8	-ie9	-ie14	-ie15	ie12	ie13	-ie2	ie3	-i1	-ie1	-ie6	-ie7	ie4	ie5	-ee10	-ee11	-ee8	ee9	ee14	ee15	-ee12	-ee13	ee2	-ee3	e1	ee1	ee6	ee7	-ee4	-ee5
ie11	-ie10	ie9	ie8	-ie15	ie14	-ie13	ie12	-ie3	-ie2	ie1	-i1	-ie7	ie6	-ie5	ie4	-ee11	ee10	-ee9	-ee8	ee15	-ee14	ee13	-ee12	ee3	ee2	-ee1	e1	ee7	-ee6	ee5	-ee4
ie12	ie13	ie14	ie15	ie8	-ie9	-ie10	-ie11	-ie4	ie5	ie6	ie7	-i1	-ie1	-ie2	-ie3	-ee12	-ee13	-ee14	-ee15	-ee8	ee9	ee10	ee11	ee4	-ee5	-ee6	-ee7	e1	ee1	ee2	ee3
ie13	-ie12	ie15	-ie14	ie9	ie8	ie11	-ie10	-ie5	-ie4	ie7	-ie6	ie1	-i1	ie3	-ie2	-ee13	ee12	-ee15	ee14	-ee9	-ee8	-ee11	ee10	ee5	ee4	-ee7	ee6	-ee1	e1	-ee3	ee2
ie14	-ie15	-ie12	ie13	ie10	-ie11	ie8	ie9	-ie6	-ie7	-ie4	ie5	ie2	-ie3	-i1	ie1	-ee14	ee15	ee12	-ee13	-ee10	ee11	-ee8	-ee9	ee6	ee7	ee4	-ee5	-ee2	ee3	e1	-ee1
ie15	ie14	-ie13	-ie12	ie11	ie10	-ie9	ie8	-ie7	ie6	-ie5	-ie4	ie3	ie2	-ie1	-i1	-ee15	-ee14	ee13	ee12	-ee11	-ee10	ee9	-ee8	ee7	-ee6	ee5	ee4	-ee3	-ee2	ee1	e1

It was generated by means of **JHyperComplex** using a (new) **tensor algebra** class.

See also:

- *Bicomplex number*
- **Biquaternion**

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Bol Algebra

A **(Left) Bol Algebra** is a **Lie Triple system**, having in addition a binary operation (,), satisfying

$$\begin{aligned}(\mathbf{A},\mathbf{B}) &= -(\mathbf{B},\mathbf{A})\\ (\mathbf{A},\mathbf{B},(\mathbf{C},\mathbf{D})) &= ((\mathbf{A},\mathbf{B},\mathbf{C}),\mathbf{D})+(\mathbf{C},(\mathbf{A},\mathbf{B},\mathbf{D}))+(\mathbf{C},\mathbf{D},(\mathbf{A},\mathbf{B}))+((\mathbf{A},\mathbf{B}),(\mathbf{C},\mathbf{D}))\end{aligned}$$

for any $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$ of its vector space.
(Right Bol algebras can be defined in a similar fashion and do not exhibit new mathematics. One can therefore restrict the treatment to either a left or a right Bol algebra. We'll consider the left case).

For any local analytic **(left/right) Bol loop**, a structure of a **Bol algebra** can be introduced on the **tangent space** at unit in a canonical way and is called the (left/right) tangent Bol algebra.
Theorem
Any Bol algebra is isomorphic to a tangent Bol algebra, associated uniquely to some **local analytic Bol loop**.

Bol algebras generalise **Lie algebras** and **Malcev algebras**.

If the trilinear operation in the above definition vanishes identically then the definition becomes that of a Lie algebra.

In any Malcev algebra a ternary bracket can be defined by

$$(\mathbf{A},\mathbf{B},\mathbf{C}) = ((\mathbf{A},\mathbf{B}),\mathbf{C}) - \frac{1}{3}\,\mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{C})$$

Papers:

- The Representation of Bol Algebras (2003) - Ndoune, T. B. Bouetou local pct. 0
- Sabinin's Method for Classification of Local Bol Loops (1999) - A. Vanžurová local pct. 0
- On the Structure of Bol Algebras (2003) - T. B. Bouetou local pct. 0

Documents:

- Proceedings of the Eighteenth Annual University-wide Seminar WORKSHOP 2009 at the Czech Technical University in Prague local

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Bol Loop

A **Left Bol Loop** satisfies the identity

$$\begin{aligned}(\mathbf{A}(\mathbf{BA}))\mathbf{C} &= \mathbf{A}(\mathbf{B}(\mathbf{AC})) \Leftrightarrow \\ L_{\mathbf{A}\cdot\mathbf{BA}} &= L_{\mathbf{A}} L_{\mathbf{B}} L_{\mathbf{A}}\,,\end{aligned}$$

a **Right Bol Loop** the identity

$$\begin{aligned}((\mathbf{AB})\mathbf{C})\mathbf{B} &= \mathbf{A}((\mathbf{BC})\mathbf{B}) \Leftrightarrow \\ R_{\mathbf{B}} R_{\mathbf{C}} R_{\mathbf{B}} &= R_{\mathbf{BC}\cdot\mathbf{B}}\end{aligned}$$

The two classes of loops are closely related to one another and the mathematical treatment of one of them is sufficient. We'll consider the left Bol loops henceforward.

Loops obeying either the left or the right Bol identity are **power-associative**.

A loop is both left- and right Bol if and only if it is a **Moufang loop**.
Consequently, any Moufang loop is a Bol loop. The smallest order of a Bol loop (which is not Moufang) is eight. There are six such loops.

A Bol loop is **diassociative** if and only if it is a Moufang loop.

Related to Bol Loops are **Bol algebras**.

Generalisations

- M-loops and PL-loops generalize smooth Bol loops.
- Hyporeductive loops are a generalization both of Bol loops and homogeneous loops. These are related to hyporeductive spaces.

Papers:

- Bol Loops (1965) - D. A. Robinson local pct. 74
- A Class of Simple Proper Bol Loops (2007) - G. P. Nagy local pct. 10
- A Brief History of Loop Rings (1998) - E. G. Goodaire local pct. 1

Links:

- WIKIPEDIA - Gerrit Bol
- Finite Simple Left Bol Loops by Gábor P. Nagy

Google Books:

- Loops in Group and Lie Theory - P. T. Nagy, K. Strambach local bct. 22

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Bruck Loop

A **(Left) Bruck Loop** or **K-Loop** is a **(left) Bol loop**, satisfying the **Left Bruck Identity**

$$(\mathbf{AB})(\mathbf{AB}) = \mathbf{A}(\mathbf{B}(\mathbf{BA}))$$

or equivalently the **Automorphic Inverse Identity**

$$(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$$

Left Bruck loops are equivalent to Ungar's **gyrocommutative gyrogroups**.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Canonical Expansion

Given a **quasigroup** $\mathcal{Q}$ and elements $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$ (infinitesimally) close to the identity $\mathbf{e}$, the operation of quasigroup multiplication can be transferred to the **tangent space** via the *exponential map* and is given by

$$\begin{aligned}\mathbf{AB} \equiv \mathbf{C} &= \exp(\boldsymbol{d}a^\mu \partial_{a^\mu}) \exp(\boldsymbol{d}b^\mu \partial_{b^\mu}) \Phi(\mathbf{x}) \\ &\equiv \exp(\boldsymbol{D}c^\mu(a^1, \ldots a^n, b^1, \ldots b^n)\partial_{c^\mu})\Phi(\mathbf{x})\end{aligned}$$

Taking into account the quasigroup property of the identity, $\mathbf{Ae} = \mathbf{A}$ and $\mathbf{eB} = \mathbf{B}$, we get
$\boldsymbol{D}c^\mu(a^1, \ldots, a^n, 0, \ldots, 0)\partial_{c^\mu} = \boldsymbol{d}a^\mu \partial_{a^\mu}$ and $\boldsymbol{D}c^\mu(0, \ldots, 0, b^1, \ldots, b^n)\partial_{c^\mu} = \boldsymbol{d}b^\mu \partial_{b^\mu}$. That is to say, the arguments of the exponentials of $\mathbf{A}$ and $\mathbf{B}$, respectively, have to be reproduced in these cases.
Using the fact that $\boldsymbol{D}c^\mu$ is given by $\boldsymbol{D}c^\mu(\mathbf{a}, \mathbf{b}) = c^\mu(\boldsymbol{d}\mathbf{a}, \boldsymbol{d}\mathbf{b}) - c^\mu(\mathbf{0}, \mathbf{0}) = c^\mu(\boldsymbol{d}\mathbf{a}, \boldsymbol{d}\mathbf{b})$ and doing a Taylor series expansion of c^μ, considering the conditions just mentioned, one gets

$$\begin{aligned} \boldsymbol{D}c^\mu(\mathbf{a},\mathbf{b}) &= \boldsymbol{da}^\mu + \boldsymbol{db}^\mu + p^\mu_{\nu\rho}\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho + \\ &\quad \frac{1}{2}\left(q^\mu_{\nu\rho\sigma}\boldsymbol{da}^\nu \otimes \boldsymbol{da}^\rho \otimes \boldsymbol{db}^\sigma + r^\mu_{\nu\rho\sigma}\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho \otimes \boldsymbol{db}^\sigma\right) + \\ &\quad \frac{1}{3!}\, s^\mu_{\nu\rho\sigma\tau}\boldsymbol{da}^\nu \otimes \boldsymbol{da}^\rho \otimes \boldsymbol{da}^\sigma \otimes \boldsymbol{db}^\tau + \frac{1}{2\cdot 2}\, t^\mu_{\nu\rho\sigma\tau}\boldsymbol{da}^\nu \otimes \boldsymbol{da}^\rho \otimes \boldsymbol{db}^\sigma \otimes \boldsymbol{db}^\tau + \frac{1}{3!}\, u^\mu_{\nu\rho\sigma\tau}\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho \otimes \boldsymbol{db}^\sigma \otimes \boldsymbol{db}^\tau + \\ &\quad \sum_{s=5}^{\infty} \mathfrak{p}_g(\{\boldsymbol{da},\boldsymbol{db}\}) \\ &\equiv \sum_{g=1}^{\infty} \boldsymbol{d}_g c^\mu(\mathbf{a},\mathbf{b}) \equiv \bigoplus_{g=1}^{\infty} \boldsymbol{d}_g c^\mu(\mathbf{a},\mathbf{b}) \\ &\equiv \sum_{g=1}^{\infty} \mathfrak{p}_s(\{\boldsymbol{da},\boldsymbol{db}\}) \end{aligned}$$

with $\mathfrak{p}_g$ monomials of order g in $\boldsymbol{da}^\nu$ and $\boldsymbol{db}^\rho$, ($\nu,\rho \in \{1,\ldots,n\}$). g corresponds with a grade of the tangent space, or alternatively speaking, with the order of infinitesimal changes.

The first few coefficients of the expansion satisfy the obvious symmetries:
$q^\mu_{\nu\rho\sigma} = q^\mu_{\rho\nu\sigma}$, $r^\mu_{\nu\rho\sigma} = r^\mu_{\nu\sigma\rho}$, $s^\mu_{\nu\rho\sigma\tau} = s^\mu_{\sigma(\nu)\sigma(\rho)\sigma(\sigma)\tau}$, $t^\mu_{\nu\rho\sigma\tau} = t^\mu_{\rho\nu\sigma\tau} = t^\mu_{\nu\rho\tau\sigma} = t^\mu_{\rho\nu\tau\sigma}$ and $u^\mu_{\nu\rho\sigma\tau} = u^\mu_{\nu\sigma(\rho)\sigma(\sigma)\sigma(\tau)}$.

Choosing **canonical coordinates**, one has $c^\mu(\mathbf{a},\mathbf{a}) = 2\mathbf{a}$.
This means that all terms of order 2 and higher have to vanish, leading to further symmetries of the coefficients (cyclic ones), given by

$$\begin{aligned} p^\mu_{(\nu\rho)} &= p^\mu_{\nu\rho} + p^\mu_{\rho\nu} = 0 \\ q^\mu_{(\nu\rho\sigma)} + r^\mu_{(\nu\rho\sigma)} &= +\, q^\mu_{\nu\rho\sigma} + q^\mu_{\nu\sigma\rho} + q^\mu_{\rho\nu\sigma} + q^\mu_{\rho\sigma\nu} + q^\mu_{\sigma\nu\rho} + q^\mu_{\sigma\rho\nu} \\ &\quad +\, r^\mu_{\nu\rho\sigma} + r^\mu_{\nu\sigma\rho} + r^\mu_{\rho\nu\sigma} + r^\mu_{\rho\sigma\nu} + r^\mu_{\sigma\nu\rho} + r^\mu_{\sigma\rho\nu} = 0 \\ s^\mu_{(\nu\rho\sigma\tau)} + t^\mu_{(\nu\rho\sigma\tau)} + u^\mu_{(\nu\rho\sigma\tau)} &= 0 \quad (72 \text{ terms}) \\ &\ldots \end{aligned}$$

An expansion satisfying these symmetries is called **Canonical Expansion**.

Algebraic realisation

The coefficients depend on the quasigroup in regards (E.g. for a **Bol loop** which is a related to a ternary-binary tangent algebra they are different than for a **group**, which is related to a binary tangent algebra).

To establish the relationship between algebra and differential geometry (i.e. the manifold and the tangent space), we consider algebraic objects, which correspond to certain orders of the canonical expansion, e.g. the **commutator** and the **associator**.

Commutator expansion

For the commutator we get, applying the series expansion of the exponential function,

$$\begin{aligned} [\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA} &= \exp(\partial_\mu \boldsymbol{da}^\mu + \partial_\mu \boldsymbol{db}^\mu + p_{\nu\rho}\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho + \mathcal{O}(3)) - \exp(\partial_\mu \boldsymbol{db}^\mu + \partial_\mu \boldsymbol{da}^\mu + p_{\nu\rho}\boldsymbol{db}^\nu \otimes \boldsymbol{da}^\rho + \mathcal{O}(3)) \\ &= \frac{1}{2}\, p_{\nu\rho}(\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho - \boldsymbol{db}^\nu \otimes \boldsymbol{da}^\rho) + \mathcal{O}(3) \\ &= p_{\nu\rho}\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho + \mathcal{O}(3) \\ &= p^\mu_{\nu\rho}\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho \otimes \mathbf{e}_\mu + \mathcal{O}(3) \\ &\equiv T^\mu(\mathbf{e}_\nu,\mathbf{e}_\rho)\boldsymbol{da}^\nu \otimes \boldsymbol{db}^\rho \otimes \mathbf{e}_\mu + \mathcal{O}(3) \end{aligned}$$

where we have used the antisymmetry of the $p^\mu_{\nu\rho}$ from above.

I.e. the coefficients $p^\mu_{\nu\rho}$ of the second order of the canonical expansion are identical with the coefficients of the **torsion tensor**.
Hence these coefficients are tensorial in nature and we could as well have chosen another basis.
In fact it can be proved that the canonical expansions of the equations of a local analytic quasigroup preserve their form if and only if the variables appearing in them are transformed by means of the formulas

$$\tilde{a}^\mu = h^\mu_\nu a^\nu, \quad \tilde{b}^\mu = h^\mu_\nu b^\nu, \quad \tilde{c}^\mu = h^\mu_\nu c^\nu$$

(The letter h is intended to allude to a possible interpretation of these transformations in terms of classical **tetrad fields**).

Associator expansion

Accordingly for the associator, yet somewhat more painful to calculate, we get

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = \left\{\frac{1}{4}\left(T^\tau_{\mu\nu}T^\sigma_{\tau\rho} - T^\sigma_{\mu\tau}T^\tau_{\nu\rho}\right) + q^\sigma_{\mu\nu\rho} - r^\sigma_{\mu\nu\rho}\right\}\boldsymbol{da}^\mu \otimes \boldsymbol{db}^\nu \otimes \boldsymbol{dc}^\rho \otimes \mathbf{e}_\sigma + \mathcal{O}(4)$$

3-webs

It can be shown [1] that for a 3-webs $p^\mu_{\nu\rho}$, $q^\mu_{\nu\rho\sigma}$ and $r^\mu_{\nu\rho\sigma}$ are given by

$$p^\mu_{\nu\rho} = \frac{1}{2}\left[T^\mu_{\nu\rho} - T^\mu_{\rho\nu}\right] = T^\mu_{[\nu\rho]} = T^\mu(\mathbf{e}_\nu,\mathbf{e}_\rho)$$

$$\begin{aligned} q^\mu_{\nu\rho\sigma} &= -\frac{1}{2}\, A^\mu_{(\nu\rho\sigma)} + \frac{4}{3}\left(A^\mu_{[\sigma|\rho|\nu]} + A^\mu_{[\sigma|\nu|\rho]} + T^\tau_{(\nu|\sigma|}T^\mu_{\rho)\tau}\right) \\ &= \frac{1}{3!}\left\{-\frac{1}{2}\left(A^\mu_{\nu\rho\sigma} + A^\mu_{\nu\sigma\rho} + A^\mu_{\rho\nu\sigma} + A^\mu_{\rho\sigma\nu} + A^\mu_{\sigma\nu\rho} + A^\mu_{\sigma\rho\nu}\right) + \right. \\ &\quad \left. 4\left(A^\mu_{\sigma\rho\nu} + A^\mu_{\sigma\nu\rho} - A^\mu_{\nu\rho\sigma} - A^\mu_{\rho\nu\sigma}\right) + 4\left(T^\tau_{\rho\sigma}T^\mu_{\nu\tau} + T^\tau_{\nu\sigma}T^\mu_{\rho\tau}\right)\right\} \end{aligned}$$

and

$$\begin{aligned} r^{\mu}_{\nu\rho\sigma} &= \frac{1}{2}\, A^{\mu}_{(\nu\rho\sigma)} + \frac{4}{3}\, \left(A^{\mu}_{[\rho\nu]\sigma} + A^{\mu}_{[\sigma\nu]\rho} + T^{\tau}_{(\rho|\nu|} T^{\mu}_{\sigma)\tau} \right) \\ &= \frac{1}{3!} \Big\{ \frac{1}{2} \left(A^{\mu}_{\nu\rho\sigma} + A^{\mu}_{\nu\sigma\rho} + A^{\mu}_{\rho\nu\sigma} + A^{\mu}_{\rho\sigma\nu} + A^{\mu}_{\sigma\nu\rho} + A^{\mu}_{\sigma\rho\nu} \right) + \\ & 4 \left(A^{\mu}_{\rho\nu\sigma} + A^{\mu}_{\sigma\nu\rho} - A^{\mu}_{\nu\rho\sigma} - A^{\mu}_{\nu\sigma\rho} \right) + 4 \left(T^{\tau}_{\rho\sigma} T^{\mu}_{\nu\tau} + T^{\tau}_{\nu\sigma} T^{\mu}_{\rho\tau} \right) \Big\} \end{aligned}$$

with $T^{\mu}_{\nu\rho}$ the **torsion tensor** and $A^{\mu}_{\nu\rho\sigma}$ the **nonassociativity tensor**.

Google books:

- [1] Geometry and Algebra of Multidimensional Three-webs - M. A. Akivis, A. M. Shelekhov local bct. 60 prl. 10

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Cayley-Dickson Algebra

> The Cayley-Dickson algebras are just Clifford algebras with some signs in their multiplication tables changed, to make the elements all mutually anticommute. The two lines of algebras both agree on the reals, the complex numbers and the quaternions, but then they part company: Clifford goes next to an 8-dimensional matrix algebra while Cayley-Dickson produces the 8-dimensional octonions.
> - D. R. Finkelstein - Time, Quantum, and Information

Construction

A **Cayley-Dickson (CD) Algebra** can be generated by means of the (classical) **Cayley-Dickson doubling process**.
The sequence $\mathbb{A}_n$ of algebras generated is: $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$, $\mathbb{A}_3 = \mathbb{O}$, $\mathbb{A}_4 = \mathbb{S}$, $\mathbb{A}_5 = \mathbb{T}, \ldots$ namely the real numbers, complex numbers, **quaternions**, **octonions**, **sedenions**, **trigintaduonions**,....

All Cayley-Dickson- and **Clifford-algebras** can be obtained by so called proper twists on $\mathbb{Z}_2^N$ (for details see [1]).

General Properties

$\mathbb{A}_0$ and $\mathbb{A}_1$ are commutative.
$\mathbb{A}_0$, $\mathbb{A}_1$ and $\mathbb{A}_2$ are associative.
$\mathbb{A}_0, \ldots, \mathbb{A}_3$ are **alternative** and **normed**.
$\mathbb{A}_n$ with $n \geq 4$ is **flexible** and has **zero divisors**.

Given a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2^n}\}$ of $\mathbb{A}_n$, one has $\prod_{i=1}^{2^n} \mathbf{e}_i = \pm 1$. The result being $+1$ or -1 depends on the order of the multiplications and the **association type** of the $2^n - 1$-fold product.

Conjugation

Due to flexibility, the conjugacy map $\mathbf{XAX}^*$ is well defined for all CD-algebras.
Proof:
Assume

$$\mathbf{X}(\mathbf{AX}^*) = (\mathbf{XA})\mathbf{X}^* \equiv \mathbf{XAX}^*$$

Due to $\mathbf{X} + \mathbf{X}^* = \text{const.}$

$$\begin{aligned} \mathbf{X}(\mathbf{A}(-\mathbf{X} + \text{const.})) &= (\mathbf{XA})(-\mathbf{X} + \text{const.}) \Longleftrightarrow \\ -\mathbf{X}(\mathbf{AX}) + \text{const.}\mathbf{AX} &= -(\mathbf{XA})\mathbf{X} + \text{const.} \end{aligned}$$

Using flexibility, equivalence follows.

Automorphism groups

Given CD algebras $\mathbb{A}_n = CD(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i = \pm 1$, obtained by a classical Cayley-Dickson doubling process, one finds the following automorphism groups:

- $Aut(\mathbb{A}_1)$ = $\mathbb{Z}_2$ = {Identity, Conjugation}.
- $Aut(\mathbb{A}_2)$ = $SO(3)$, the rotation group in $\mathbb{R}^3$.
- $Aut(\mathbb{A}_3)$ = **G_2**, the exceptional **Lie group**, if the octonion algebra is non-split. In the split case, the automorphism group is **split G_2**, $\tilde{G}_2$. As all **split octonion algebras** are **isomorphic**, there are only these two cases.
- $Aut(\mathbb{A}_n)$ = $Aut(\mathbb{A}_{n-1}) \times S_3$ if $\lambda_n = 1$ and $Aut(\mathbb{A}_n)$ = $Aut(\mathbb{A}_{n-1}) \times S_2$ if $\lambda_n = -1$ for $n \geq 4$ (shown by R. B. Brown [2] and P. Eakin and A. Sathaye [3]), where S_3 and S_2 are the **symmetric groups** of **order** 6 and 2 respectively.

Therefore $Aut(\mathbb{A}_4) = G_2 \times S_3$, $G_2 \times S_2$, $\tilde{G}_2 \times S_3$ or $\tilde{G}_2 \times S_2$, depending on the doubling process.

Adjoint properties

All Cayley-Dickson algebras satisfy the adjoint properties which they have in common with **Clifford algebras**.
For a *pure* operator $\mathbf{D}$ one gets

$$\langle \boldsymbol{D\Psi'}|\boldsymbol{\Psi}\rangle = -\langle \boldsymbol{\Psi'}|\boldsymbol{D\Psi}\rangle$$

and

$$\langle \boldsymbol{\Psi' D}|\boldsymbol{\Psi}\rangle = -\langle \boldsymbol{\Psi'}|\boldsymbol{\Psi D}\rangle$$

Thus $\mathbf{D}$ is **anti-hermitian**. This suggests a formulation of **quantum mechanics based on Cayley-Dickson algebras**.

Norm properties

$$||\mathbf{AB}|| = ||\mathbf{A^*B}|| = ||\mathbf{AB^*}|| = ||\mathbf{BA}||$$

$||\mathbf{AB}|| = ||\mathbf{BA}||$ is also known as **Weakened Norm-Multiplicativity/Commutativity Property**.
For the algebras $\mathbb{A}_n$ with $n > 3$ one has

$$||\mathbf{AB}|| \leq \left(\sqrt{2}\right)^{n-3} ||\mathbf{A}|| \cdot ||\mathbf{B}||$$

If $n \leq 3$ and $\mathbb{A}_n$ is non-split, i.e. $\mathbb{A}_n$ doesn't contain zero-divisors, one has equality, as these algebras are **composition algebras**.

Associator-identities

$$-[\mathbf{A},\mathbf{B},\mathbf{C}] = [\mathbf{A}^*,\mathbf{B},\mathbf{C}] = [\mathbf{A},\mathbf{B}^*,\mathbf{C}] = [\mathbf{A},\mathbf{B},\mathbf{C}^*]$$

Rotations

The multiplication map $R_{\mathbf{B}} : \mathbb{A}_n \to \mathbb{A}_n : \mathbf{A} \mapsto \mathbf{AB}$ with $||\mathbf{B}|| = 1$ belongs to $SO(2^n)$ for $n \leq 3$ as these algebras are **composition algebras**.
Therefore:
$(R_{\mathbf{B}\in\mathbb{C}} : \mathbb{A}_1 \to \mathbb{A}_1) \in SO(2)$
$(R_{\mathbf{B}\in\mathbb{H}} : \mathbb{A}_2 \to \mathbb{A}_2) \in$ *SO(4)*
$(R_{\mathbf{B}\in\mathbb{O}} : \mathbb{A}_3 \to \mathbb{A}_3) \in$ **SO(8)**

Zero Divisors

In the following we only consider zero divisors of the non-split algebras obtained by a standard **Cayley-Dickson doubling process**. (In regards to the split case, see **split algebra zero divisor**).

- Every zero-divisor in $\mathbb{A}_n$ is imaginary.
- For the dimension d of zero divisor subspaces in $\mathbb{A}_n$ one has: $d \leq 2^n - 4n + 4$. d is always a multiple of 4 and for any possible dimension, zero divisors do in fact occur. Therefore in the **sedenions** one has only zero divisors for $d = 0, 4$, whereas in the **trigintaduonions** they occur for $d = 0, 4, 8, 16$.
- For $n \geq 4$ the largest zero divisor subspace of $\mathbb{A}_n$ is **homeomorphic** to a disjoint union of 2^{n-4} copies of the **Stiefel variety** $V_2(\mathbb{R}^7)$, i.e. the space of ordered pairs of orthonormal vectors in $\mathbb{R}^7$. For the sedenions one therefore gets one copy (see also **sedenion zero divisor**) and for the trigintaduonions 2 copies.

Alternative Subspaces

Theorem The greatest possible dimension d of an alternative subspace of $\mathbb{A}_n$ is $d = 2(n+1)$, given $n \geq 3$. Hence $d_{\mathbb{S}} = 10$ and $d_{\mathbb{T}} = 12$.

Multiplication Tables

For the number of inequivalent multiplication tables $N(\mathbb{A}_n)$ of $\mathbb{A}_n$, the conjecture is (see also [4])

$$\begin{aligned} N(\mathbb{A}_n) &= N_{Fano}(\mathbb{A}_n) \cdot N_{signs} \\ &= \frac{(2^n-1)!}{\mathrm{ord}(PGL(n,2))} \cdot 2^{2^n-n-1} \end{aligned}$$

where $N_{Fano}(A_n)$ is the number of inequivalent **Fano spaces** associated with the algebra and N_{signs} the number of sign permutations of base vectors leading to inequivalent multiplication tables. $\mathrm{ord}(PGL(n,2))$ is the order of the **projective linear group** $PGL(n,2)$.

Thence:

$$\begin{aligned} N(\mathbb{H}) &= N_{Fano\,lines}(\mathbb{H}) \cdot N_{signs}(\mathbb{H}) = 1 \cdot 2 = 2 \\ N(\mathbb{O}) &= N_{Fano\,planes}(\mathbb{O}) \cdot N_{signs}(\mathbb{O}) = \frac{7!}{168} \cdot 2^4 = 30 \cdot 16 = 480 \\ N(\mathbb{S}) &= N_{Fano\,tetrahedra}(\mathbb{S}) \cdot N_{signs}(\mathbb{S}) = \frac{15!}{20.160} \cdot 2^{11} = 64.864.800 \cdot 2.048 = 132.843.110.400 \approx 1.3 \cdot 10^{11} \\ N(\mathbb{T}) &= N_{Fano\,hyper-tetrahedra}(\mathbb{T}) \cdot N_{signs}(\mathbb{T}) = \frac{31!}{9.999.360} \cdot 2^{26} = 822.336.494.953.469.303.808.000.000 \cdot 67.108.864 \approx 5.5 \cdot 10^{34} \\ N(\mathbb{A}_6) &= N_{Fano\,hyper^2-tetrahedra}(\mathbb{A}_6) \cdot N_{signs}(\mathbb{A}_6) = \frac{63!}{20.158.709.760} \cdot 2^{57} \approx 1.4 \cdot 10^{94} \end{aligned}$$

Relevance

Cayley-Dickson algebras are at the core of mathematics, among other things as they are part of the algebraic backbone of the **exceptional** and **sporadic** structures. It would therefore only be surprising if higher order algebras (octonions and onwards) play no role in physics.

Papers:

- [2] On Generalized Cayley-Dickson Algebras (1967) - R. B. Brown local pct. 30 prl. 10
- Functions of Several Cayley-Dickson Variables and Manifolds over them (2006) - S. V. Lüdkovsky local pct. 23 TRD
- Alternative Elements in the Cayley-Dickson Algebras (2004) - G. Moreno local pct. 6
- Quantum Exotic PDE's (2012) - A. Prástaro local pct. 4
- [1] Properly Twisted Groups and their Algebras (2006) - J. W. Bales local pct. 2 prl. 9
- [4] The 42 Assessors and the Box-Kites they fly: Diagonal Axis-Pair Systems of Zero-Divisors in the Sedenions' 16 Dimensions (2001) - R. P. C. de Marrais local pct. 2
- Cayley-Dickson and Clifford Algebras as Twisted Group Algebras (2003) J. W. Bales local pct. 0
- A Tree for Computing the Cayley-Dickson Twist (2000) - J. W. Bales local pct. 0

Links:

- The Cayley-Dickson Calculator

Journals:

- [3] On Automorphisms and Derivations of Cayley-Dickson Algebras (1990) - P. Eakin, A. Sathaye jct. 16

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Cayley-Dickson Doubling

> The Cayley-Dickson Process, as its name suggests, is due to A. A. Albert.
> - [1], see also *Stigler's law of eponymy* -

The **Cayley-Dickson Doubling (Process)** (a.k.a. **Cayley-Dickson Construction**) doubles a **Cayley-Dickson algebra**, extending its multiplication, **involution** * and **norm** $||\,.\,||$ to the new Cayley-Dickson algebra. The resulting algebra contains the originial algebra as a subalgebra.

Two common and equivalent formulas for the **Extended Multiplication** are

$$(\mathbf{A}_1,\mathbf{A}_2)(\mathbf{B}_1,\mathbf{B}_2)=(\mathbf{A}_1\mathbf{B}_1-\lambda\mathbf{B}_2^*\mathbf{A}_2,\mathbf{B}_2\mathbf{A}_1+\mathbf{A}_2\mathbf{B}_1^*)$$

and

$$(\mathbf{A}_1,\mathbf{A}_2)(\mathbf{B}_1,\mathbf{B}_2)=(\mathbf{A}_1\mathbf{B}_1-\lambda\mathbf{B}_2\mathbf{A}_2^*,\mathbf{B}_1\mathbf{A}_2+\mathbf{A}_1^*\mathbf{B}_2)$$

where $\lambda=\pm1$. (Notice that the sign of λ is a matter of convention and formulas with a "+λ"-term are also found in literature).
We'll (try to) stick to the first formula.

For the **Extended Involution** one has

$$(\mathbf{A},\mathbf{B})^*=(\mathbf{A}^*,-\mathbf{B})$$

and for the **Extended Norm** $||\,.\,||$

$$||(\mathbf{A},\mathbf{B})||^2=||\mathbf{A}||^2+\lambda||\mathbf{B}||^2$$

A Cayley-Dickson algebra that results from n doubling steps will be designated $CD(\lambda_1,\lambda_2,\ldots,\lambda_n)$ with $\lambda_i=\pm1$.
If $\lambda_i=1,\ \forall i$, one gets a (standard/canonical) **Non-split (Circular) Cayley-Dickson Algebra**. Else the algebras are called ***Split (Hyperbolic)*** **Cayley Dickson Algebras**.

In case of $n\leq3$ one has the theorem, that any two split **composition algebras** of the same dimension over a given field are **isomorphic**. The proof is based on the **Moufang identity** and thus doesn't go through for higher order algebras. In fact these exhibit non-isomorphic split algebras.

The *multiplication tables* resulting from doublings with different λ's differ only in the signs of their *structure constants* (see for instance, **octonion multiplication tables**). In particular, the structure of the multiplication tables of split and non-split algebras differs only in respect to their associated **sign tables** and not to their underlying **Fano spaces**.

A $N=2^n$-dimensional split-algebra has a multiplication table with **signature** $(\frac{N}{2}+1,\frac{N}{2}-1)$.
For the split versions of the following Cayley-Dickson algebras one therefore has the signatures:

- *Complex numbers*: $(2,0)$
- **Quaternions**: $(3,1)$
- **Octonions**: $(5,3)$
- **Sedenions**: $(9,7)$
- Trigintaduonions: $(17,15)$

The signs of the N terms of the square of the norm of these algebras consists of an equal number of "+" and "-" signs (i.e. $N/2$ of each sign). (Is this the reason why these algebras are called split algebras ?)

Notice, that the Cayley-Dickson doubling formula is limited in that it can, the signature being fixed, only generate one algebra of given dimension n. However from dimension $n=2$ onwards there exist several different algebras (e.g. 2, 480, $132.843.110.400$ in case of the **quaternions**, **octonions** and **sedenions**). Therefore a more general construction which allows one to get all of them would be desirable.

Alternative Formulas

2n-ons

Computer searches by Warrent Smith have yielded over 100 different doubling formulas beyond the octonions, he calls 2^n-ons, with all kind of different algebraic properties, most of them however "unpleasant".

Twisted Cayley-Dickson algebras

D. Chesley has considered modifications of the CD doubling process quite generically. Ignoring the split algebras, he comes up with 512 possible forms to consider. All but 32 are eliminated by not satisfying $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$. Of the 32 remaining, 2 are the classical formulas described above, 28 produce multiplication tables in which some triad does not have quaternionic properties, and 2 new variants emerge, which are related to each other in the same way that the usual two Cayley-Dickson forms are related to each other. These he calls *twisted Cayley-Dickson algebras*.

CD-like products

Following [1], an extended product is called **Cayley-Dickson-like**, if it satisfies

$$\begin{aligned}(\mathbf{e},\mathbf{0})(\mathbf{B}_1,\mathbf{B}_2) &= (\mathbf{B}_1,\mathbf{B}_2)\\ (\mathbf{A}_1,\mathbf{A}_2)(\mathbf{e},\mathbf{0}) &= (\mathbf{A}_1,\mathbf{A}_2)\\ (\mathbf{A}_1,\mathbf{A}_2)(\mathbf{A}_1^*,-\mathbf{A}_2) &= \left(||\mathbf{A}_1||^2+|||\mathbf{A}_2||^2,\mathbf{0}\right)\\ (\mathbf{A}_1^*,-\mathbf{A}_2)(\mathbf{A}_1,\mathbf{A}_2) &= \left(||\mathbf{A}_1||^2+|||\mathbf{A}_2||^2,\mathbf{0}\right)\end{aligned}$$

On the level of the extended algebra this reads

$$\begin{aligned}\mathbf{eC} &= \mathbf{Ce}=\mathbf{C}\\ \mathbf{CC}^* &= \mathbf{C}^*\mathbf{C}=||\mathbf{C}||^2\end{aligned}$$

32 different such products can be found which are as follows:

$$\begin{aligned}
&P_0:(a,b)(c,d)=(ca-b^*d,da^*+bc)\\
&P_1:(a,b)(c,d)=(ca-db^*,da^*+bc)\\
&P_2:(a,b)(c,d)=(ca-b^*d,a^*d+cb)\\
&P_3:(a,b)(c,d)=(ca-db^*,a^*d+cb)\\
&P_4:(a,b)(c,d)=(ac-b^*d,da^*+bc)\\
&P_5:(a,b)(c,d)=(ac-db^*,da^*+bc)\\
&P_6:(a,b)(c,d)=(ac-b^*d,a^*d+cb)\\
&P_7:(a,b)(c,d)=(ac-db^*,a^*d+cb)\\
&P_8:(a,b)(c,d)=(ca-bd^*,da^*+bc)\\
&P_9:(a,b)(c,d)=(ca-d^*b,da^*+bc)\\
&P_{10}:(a,b)(c,d)=(ca-bd^*,a^*d+cb)\\
&P_{11}:(a,b)(c,d)=(ca-d^*b,a^*d+cb)\\
&P_{12}:(a,b)(c,d)=(ac-bd^*,da^*+bc)\\
&P_{13}:(a,b)(c,d)=(ac-d^*b,da^*+bc)\\
&P_{14}:(a,b)(c,d)=(ac-bd^*,a^*d+cb)\\
&P_{15}:(a,b)(c,d)=(ac-d^*b,a^*d+cb)\\
&P_{16}:(a,b)(c,d)=(ca-b^*d,ad+c^*b)\\
&P_{17}:(a,b)(c,d)=(ca-db^*,ad+c^*b)\\
&P_{18}:(a,b)(c,d)=(ca-b^*d,da+bc^*)\\
&P_{19}:(a,b)(c,d)=(ca-db^*,da+bc^*)\\
&P_{20}:(a,b)(c,d)=(ac-b^*d,ad+c^*b)\\
&P_{21}:(a,b)(c,d)=(ac-db^*,ad+c^*b)\\
&P_{22}:(a,b)(c,d)=(ac-b^*d,da+bc^*)\\
&P_{23}:(a,b)(c,d)=(ac-db^*,da+bc^*)\\
&P_{24}:(a,b)(c,d)=(ca-bd^*,ad+c^*b)\\
&P_{25}:(a,b)(c,d)=(ca-d^*b,ad+c^*b)\\
&P_{26}:(a,b)(c,d)=(ca-bd^*,da+bc^*)\\
&P_{27}:(a,b)(c,d)=(ca-d^*b,da+bc^*)\\
&P_{28}:(a,b)(c,d)=(ac-bd^*,ad+c^*b)\\
&P_{29}:(a,b)(c,d)=(ac-d^*b,ad+c^*b)\\
&P_{30}:(a,b)(c,d)=(ac-bd^*,da+bc^*)\\
&P_{31}:(a,b)(c,d)=(ac-d^*b,da+bc^*)
\end{aligned}$$

If in addition the quaternion property holds, a product is called a Cayley-Dickson product. This means that if i,j,k are positive integers and $\mathbf{e}_i\mathbf{e}_j=\mathbf{e}_k$, then

$$\begin{aligned}\mathbf{e}_j\mathbf{e}_i &= -\mathbf{e}_k\\ \mathbf{e}_j\mathbf{e}_k &= \mathbf{e}_i\end{aligned}$$

The quaternionis property is satisfied by 8 of the 32 products, which are $P_0,P_3,P_4,P_7,P_{24},P_{27},P_{28}$ and P_{31}.
P_{31} and P_7 are the two "classical" Cayley Dickson products, given above.
P_4 and P_{28} are the two twisted products defined by D. Chesley.

(Meanwhile I have implemented the 32 doubling formulas in **JHyperComplex**. A first run yielded that P_0, P_7, P_{24} and P_{31} are **Moufang loops**. Further results will follow).

Tripled algebras

In [2] a tripling process is described which generalizes the classical Cayley-Dickson doubling process, leading to **flexible algebras**. It is given by the multiplication rule

$$(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3)=(\mathbf{A}_1\mathbf{B}_1-\lambda\mathbf{B}_2\mathbf{A}_2^*-\mu\mathbf{B}_3\mathbf{A}_3^*,\mathbf{A}_2\mathbf{B}_1+\mathbf{A}_1^*\mathbf{B}_2,\mathbf{A}_3\mathbf{B}_1+\mathbf{A}_1^*\mathbf{B}_3)$$

Upon setting $\{\mathbf{A}_2,\mathbf{B}_2\}=\mathbf{0}$ or $\{\mathbf{A}_3,\mathbf{B}_3\}=\mathbf{0}$ one reobtains the classical CD doubling formula (the second one above).

Cochain description

The Cayley-Dickson construction can equivalently be formulated by means of 2-cochains. This formulation allows for a straightforward generalisation (for details see e.g. [3]).

Clifford algebras

A doubling process for Clifford algebras and its close relationship to the Cayley-Dickson process is described in [2].

Papers:

- [3] New Approach to Octonions and Cayley Algebras (1998) - H. Albuquerque, S. Majid local pct. 4 prl. 10
- [1] A Catalog of Cayley-Dickson-like Products (2011) - J. W. Bales - J. W. Bales local pct. 0
- [2] Alternative Twisted Tensor Products and Cayley Algebras (2010) - H. Albuquerque, F. Panaite local pct. 0

Documents:

- Matters Computational - Ideas, Algorithms, Source Code (2010) - J. Arndt local - in particular: "39.14.1 The Cayley-Dickson Construction".

Links:

- WIKIPEDIA - Cayley-Dickson Construction
- [1] Kevin McCrimmon's Pre-Book on Alternative Algebras

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Center

The **Center** $Z(\mathcal{G})$ of a **group** $\mathcal{G}$ is the subset of all elements of the group that commute with all other elements of the group.

$$Z(\mathcal{G}) = \{z \in \mathcal{G} \,|gz = zg, \, \forall g \in \mathcal{G}\}$$

$Z(G)$ is an abelian subgroup of G.

Therefore the center of a group is the union of **centralizers** $C_{\mathcal{G}}(a)$ of elements a that commute with all elements of the group. (I.e. which have centralizer of maximal order).

The quotient of a group and its center is isomorphic to the group of **inner automorphisms** of the group, thus

$$\mathcal{G}/Z(\mathcal{G}) \simeq Inn(\mathcal{G})$$

Generalizations

The center of a **non-associative algebra** is defined as the intersection of the center - defined in the same way as for a group - and the **nucleus**.

Links:

- PlanetMath.org - Centralizers in Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Centralizer

The **Centralizer** of an element a of a **group** $\mathcal{G}$ (written as $C_{\mathcal{G}}(a)$) is the set of elements of $\mathcal{G}$ which commute with a. In other words, $C_{\mathcal{G}}(a) = \{x \in \mathcal{G} : xa = ax\}$.

If $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $C_{\mathcal{H}}(a) = C_{\mathcal{G}}(a) \cap \mathcal{H}$.

Links:

- WIKIPEDIA - Centralizer and Normalizer

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Cl(8)

Due to Bott periodicity any real **Clifford algebra**, no matter how large, can be embedded in a tensor product of $Cl(8)$, i.e. $\otimes_i Cl(8)_i$.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Clifford Algebra

A (Euclidean) **Clifford Algebra** is defined by the following (symmetric) product of its base elements:

$$\boxed{\{\mathbf{e}_a, \mathbf{e}_b\} \equiv 2\eta_{ab}\mathbf{e}}$$

with η_{ab} the diagonal **metric tensor**, depending on the **signature** of the algebra.
(Some mathematicians and physicists prefer this definition with an opposite sign).

A Clifford algebra hence per definition is "equipped" with a metric (and a **scalar product**).

For a non-Euclidean basis $\{\mathbf{e}_\mu(\mathbf{x})\}$ (Clifford manifold), the right hand side has to be replaced by the non-diagonal metric tensor:

$$\frac{1}{2}\{\mathbf{e}_\mu(\mathbf{x}), \mathbf{e}_\nu(\mathbf{x})\} = \langle \mathbf{e}_\mu(\mathbf{x}) | \mathbf{e}_\nu(\mathbf{x}) \rangle \mathbf{e} = g_{\mu\nu}(\mathbf{x})\mathbf{e}$$

The two cases are related by the **vielbeins** $h^a_\mu(\mathbf{x})$ via

$$\mathbf{e}_\mu(\mathbf{x}) = h^a_\mu(\mathbf{x})\mathbf{e}_a$$

Therefore

$$\frac{1}{2}\{\mathbf{e}_\mu(\mathbf{x}), \mathbf{e}_\nu(\mathbf{x})\} = h^a_\mu(\mathbf{x}) h^b_\nu(\mathbf{x})\eta_{ab}\mathbf{e}$$

The anti-symmetric part of the Clifford product, carrying a representation of the *Lorentz group*, is given by

$$\boxed{[\mathbf{e}_a, \mathbf{e}_b] = 4\sigma_{ab}}$$

with

$$\sigma_{ab} \equiv \frac{1}{2}(\mathbf{e}_a\mathbf{e}_b - \eta_{ab}\mathbf{e})$$

Or, expressed in terms of the *structure constants* C^C_{ab} of the algebra,

$$[\mathbf{e}_a, \mathbf{e}_b] = 2C^C_{ab}\mathbf{E}_C$$

Therefore a Clifford algebra basis constitutes a **nonholonomic frame**.

Examples

Name	Representation
Complex numbers	$Cl(0,1) \simeq \mathbb{C}$
Quaternions	$Cl(0,2) \simeq \mathbb{H}$
Pauli algebra	$Cl(3,0) \simeq M_2(\mathbb{C})$
Spacetime algebra	$Cl(1,3) \simeq M_2(\mathbb{H})$
Majorana algebra	$Cl(3,1) \simeq M_4(\mathbb{R})$
Dirac algebra	$Cl(4,1) \simeq M_4(\mathbb{C}) \simeq \mathbb{C} \otimes Cl(3,1) \simeq \mathbb{C} \otimes Cl(1,3)$

As a vector space, $Cl(0,n)$ is isomorphic to $\mathbb{R}^{2^n}$.

Clifford algebras have a twist

Clifford algebras can also be understood as **Twisted Group Algebras**. This description allows for setting them in a broader algebraic context, allowing for a direct comparison with other classes of algebras (e.g. **Cayley-Dickson algebras**).
A Clifford algebra with n generators is a $(\mathbb{Z}_2)^n$-graded algebra.
The complex Clifford algebra $Cl_{\mathbb{C}}(n)$ is isomorphic to the twisted group algebras over $(\mathbb{Z}_2)^n$ with the product

$$u_{(x_1,\ldots,x_n)} \cdot u_{(y_1,\ldots,y_n)} = (-1)^{\sum_{1\le i\le j\le n} x_i y_j} u_{(x_1+y_1,\ldots,x_n+y_n)}$$

where $(x_1,\ldots,x_n)$ is a n-tuple of 0's and 1's.
The above twisting function is bilinear and therefore is a *2-cocycle* on $(\mathbb{Z}_2)^n$. All higher cocycles vanish due to the fact that Clifford algebras are associative.
The real Clifford algebras $Cl_{p,q}$ are also twisted group algebras over $(\mathbb{Z}_2)^n$, where $n = p + q$.
The twisting function in the real case contains an extra term $\sum_{1\le i\le p} x_i y_i$ corresponding to the signature.

Papers:

- Clifford Algebras Obtained by Twisting of Group Algebras (2000) - H. Albuquerque, S. Majid pct. 22
- Fundamental Automorphisms of Clifford Algebras and an Extension of Dąbrowski Pin - Groups (1999) - V. V. Varlamov local pct. 15
- Clifford Algebras, Spinors and Fundamental Interactions : Twenty Years After (2005) - R. Coquereaux pct. 4
- Arithmétique dans les Algèbres de Clifford R0,q (2008) - G. Laville, M. Paugam local pct. 0 - Sheds light on the relationship between Clifford algebras and **Cayley-Dickson algebras**.

- Clifford Algebras in General Relativity (1967) - E. A. Lord local pct. 0
- Clifford Algebra: A Case for Geometric and Ontological Unification (2009) - W. M. Kallfelz local pct. 0

Links:

- Clifford-Algebras.org - With lots of interesting papers.

Google books:

- Clifford Algebras and their Applications in Mathematical Physics Aachen 1996 - V. Dietrich, H. Habeta, G Jank local bct. 2
- Clifford Algebras and Spinor Structures - R. Ablamowicz, P. Lounesto local bct. 1

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Clifford Geometric Algebra

Pascal's triangle of the number of basis k-blades in n-dimensional space

	subspace grade k					
n	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1
...			...			

Given a multivector $\mathbf{A}$, geometric algebra makes it possible to express every rotation in the canonical form

$$\mathbf{R}\mathbf{A}\mathbf{R}^{\dagger}$$

where $\mathbf{R}$ is an even multivector (called a **Rotor**) satisfying

$$\mathbf{R}\mathbf{R}^{\dagger} = 1$$

and "†" is the operation of **Reversion**, defined by

$$(\mathbf{A}\mathbf{B})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger}$$

and

$$\langle \mathbf{A} \rangle_1^{\dagger} = \langle \mathbf{A} \rangle_1$$

Rotors form a multiplicative group called the *spin group*.

The inverse $\mathbf{R}^{-1}$ of the rotation is given by

$$\mathbf{R}^{\dagger}\mathbf{A}\mathbf{R}$$

It can be shown that every rotor can be expressed in the exponential form

$$\mathbf{R} = \pm \exp(\frac{1}{2}\mathbf{B})$$

with

$$\mathbf{R}^{\dagger} = \pm \exp(-\frac{1}{2}\mathbf{B})$$

where $\mathbf{B}$ is a bivector called the **generator** of $\mathbf{R}$.

Apparently the rotor concept not only applies to Clifford algebras, but to any algebra for which the expression $\mathbf{R}\mathbf{A}\mathbf{R}^{\dagger}$ is well defined. E.g. if one takes the operation of complex conjugation instead of reversion this is the case for **Cayley-Dickson algebras**.

Theses:

- Geometric Algebra and Covariant Methods in Physics and Cosmology (2000) - A. M. Lewis tct. 3
- Constructing a "String Theory" through the use of Geometric Algebra - M. Spelt tct. 0

Lectures:

- An Introduction to Geometric Algebra and Calculus - A. Bromborsky

Presentations:

- Physical Applications of a Generalized Clifford Calculus - W. M. Pezzaglia Jr. local

Links:

- Geometric Calculus in Fukui - E. Hitzer lrl. 9
- Homepage of Markus Maute - Geometric Algebra

Google books:

- Clifford (Geometric) Algebras with Applications in Physics, Mathematics, and Engineering (1996) - W. E. Baylis local bct. 133

Videos:

- Tutorial on Clifford's Geometric Algebra (2012) - E. Hitzer

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Code Loop

Code Loops are a class of **Moufang loops**.
They were introduced by Griess (see [1]), who used them to elucidate the construction of the **Parker loop**, that is in turn involved in the construction of the **Monster group**.

Papers:

- [1] Code Loops (1986) - R. L. Griess, Jr. local pct. 44
- Explicit Constructions of Code Loops as Centrally Twisted Products (2000) - T. Hsu local pct. 9
- Class 2 Moufang Loops, Small Frattini Moufang Loops, and Code Loops - T. Hsu local pct. 0

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Color Algebra

See also:

- **QCD**

Papers:

- A Little Color in Abstract Algebra (1982) - G. P. Wene local pct. 5
- Color Algebra in Quantum Chromodynamics (2013) - T. Ma, S. Wang local pct. 0

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Commutators of Degree 4

For degree 4 one has 5 different nested **commutators**, expressed in the following in terms of **association types** and association type numbers:

1:

$$\begin{aligned}[[[A,B],C],D] &= ((AB)C)D-((BA)C)D-(C(AB))D+(C(BA))D-D((AB)C)+D((BA)C)+D(C(AB))-D(C(BA))\\ &= 1-1-4+4-2+2+5-5\end{aligned}$$

2:

$$\begin{aligned}[A,[[B,C],D]] &= A((BC)D)-A((CB)D)-A(D(BC))+A(D(CB))-((BC)D)A+((CB)D)A+(D(BC))A-(D(CB))A\\ &= 2-2-5+5-1+1+4-4\end{aligned}$$

3:

$$\begin{aligned}[[A,B],[C,D]] &= (AB)(CD)-(AB)(DC)-(BA)(CD)+(BA)(DC)-(CD)(AB)+(DC)(AB)+(CD)(BA)-(DC)(BA)\\ &= 3-3-3+3-3+3+3-3\end{aligned}$$

4:

$$\begin{aligned}[[A,[B,C]],D] &= (A(BC))D-(A(CB))D-((BC)A)D+((CB)A)D-D(A(BC))+D(A(CB))+D((BC)A)-D((CB)A)\\ &= 4-4-1+1-5+5+2-2\end{aligned}$$

5:

$$\begin{aligned}[A,[B,[C,D]]] &= A(B(CD))-A(B(DC))-A((CD)B)+A((DC)B)-(B(CD))A+(B(DC))A+((CD)B)A-((DC)B)A\\ &= 5-5-2+2-4+4+1-1\end{aligned}$$

This makes plausible that to "build" one of the **Sagle identities**, one can equally well choose one of the commutator expressions of type 1,2,3 or 5, cycle it 4 times and "balance" them by adding a commutator expression of type 3. (This is a bit more subtle than it is in case of the **Jacobi identity**, where one merely has to balance two association types which is expressed in terms of the **associator**, playing a crucial role in this context).

One feature that is common with **degree 5 commutators** is that association types show up in pairs.

Sorting the association types according to their number and their lexicographical order, one gets:

$$\begin{aligned}1: &\quad ((AB)C)D-((BA)C)D-((BC)A)D-((BC)D)A+((CB)A)D+((CB)D)A+((CD)B)A-((DC)B)A\\ 2: &\quad A((BC)D)-A((CB)D)-A((CD)B)+A((DC)B)-D((AB)C)+D((BA)C)+D((BC)A)-D((CB)A)\\ 3: &\quad (AB)(CD)-(AB)(DC)-(BA)(CD)+(BA)(DC)-(CD)(AB)+(CD)(BA)+(DC)(AB)-(DC)(BA)\\ 4: &\quad (A(BC))D-(A(CB))D-(B(CD))A+(B(DC))A-(C(AB))D+(C(BA))D+(D(BC))A-(D(CB))A\\ 5: &\quad A(B(CD))-A(B(DC))-A(D(BC))+A(D(CB))-D(A(BC))+D(A(CB))+D(C(AB))-D(C(BA))\end{aligned}$$

Transformations

The commutator association types 1, 2, 4 and 5 can easily be transformed into one another as follows:

5 → 2

$$[\boldsymbol{\partial}_i, [\boldsymbol{\partial}_j, [\boldsymbol{\partial}_k, \boldsymbol{\partial}_l]]] = -[\boldsymbol{\partial}_i, [[\boldsymbol{\partial}_k, \boldsymbol{\partial}_l], \boldsymbol{\partial}_j]]$$

1 $\rightarrow$ 4

$$[[[\boldsymbol{\partial}_i, \boldsymbol{\partial}_j], \boldsymbol{\partial}_k], \boldsymbol{\partial}_l] = -[[\boldsymbol{\partial}_k, [\boldsymbol{\partial}_i, \boldsymbol{\partial}_j]], \boldsymbol{\partial}_l]$$

The transformations $5 \rightarrow 2$ and $1 \rightarrow 4$ are related by symmetry in that all commutators are "flipped".

2 $\rightarrow$ 4

$$[\boldsymbol{\partial}_i, [[\boldsymbol{\partial}_j, \boldsymbol{\partial}_k], \boldsymbol{\partial}_l]] = [[\boldsymbol{\partial}_k, [\boldsymbol{\partial}_j, \boldsymbol{\partial}_l]], \boldsymbol{\partial}_i]$$

This sheds some light on the seeming asymmetry of the **commutator Saglean**, which necessarily consists of a commutator association type 3, that can be combined with any 4 of the other commutation association types.
So in fact it contains all 5 possible commutator association types of degree 4, which suggests that it is the appropriate 4^{th}-order generalisation of the Jacobian in 3^{rd} order. (Yet the analogy is not perfect, as in case of the Jacobian an additional cyclic permutation is involved).

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Commutators of Degree 5

For degree 5 one has 14 different nested **commutators**, expressed in the following in terms of **association types**:

(3,1)

$$\begin{aligned}
[[[[A,B],C],D],E] &= [[(AB)C-(BA)C-C(AB)+C(BA),D],E] \\
&= [((AB)C)D-((BA)C)D-(C(AB))D+(C(BA))D-D((AB)C)+D((BA)C)+D(C(AB))-D(C(BA)),E] \\
&= (((AB)C)D)E-(((BA)C)D)E-((C(AB))D)E+((C(BA))D)E-(D((AB)C))E+(D((BA)C))E+(D(C(AB)))E-(D(C(BA)))E \\
&\quad -E(((AB)C)D)+E(((BA)C)D)+E((C(AB))D)-E((C(BA))D)+E(D((AB)C))-E(D((BA)C))-E(D(C(AB)))+E(D(C(BA))) \\
&= (3,1)-(3,1)-(21,1)+(21,1)-(12,1)+(12,1)+(3,5)-(3,5) \\
&\quad -(3,2)+(3,2)+(21,2)-(21,2)+(12,2)-(12,2)-(3,6)+(3,6)
\end{aligned}$$

(3,2)

$$\begin{aligned}
[A,[[[B,C],D],E]] &= [A,[(BC)D-(CB)D-D(BC)+D(CB),E]] \\
&= [A,((BC)D)E-((CB)D)E-(D(BC))E+(D(CB))E-E((BC)D)+E((CB)D)+E(D(BC))-E(D(CB))] \\
&= A(((BC)D)E)-A(((CB)D)E)-A((D(BC))E)+A((D(CB))E)-A(E((BC)D))+A(E((CB)D))+A(E(D(BC)))-A(E(D(CB))) \\
&\quad -(((BC)D)E)A+(((CB)D)E)A+((D(BC))E)A-((D(CB))E)A+(E((BC)D))A-(E((CB)D))A-(E(D(BC)))A+(E(D(CB)))A \\
&= (3,2)-(3,2)-(21,2)+(21,2)-(12,2)+(12,2)+(3,6)-(3,6) \\
&\quad -(3,1)+(3,1)+(21,1)-(21,1)+(12,1)-(12,1)-(3,5)+(3,5)
\end{aligned}$$

(3,3)

$$\begin{aligned}
[[A,B],[[C,D],E]] &= [(AB-BA),(CD)E-(DC)E-E(CD)+E(DC)] \\
&= (AB)((CD)E)-(AB)((DC)E)-(AB)(E(CD))+(AB)(E(DC))-(BA)((CD)E)+(BA)((DC)E)+(BA)(E(CD))-(BA)(E(DC)) \\
&\quad -((CD)E)(AB)+((DC)E)(AB)+(E(CD))(AB)-(E(DC))(AB)+((CD)E)(BA)-((DC)E)(BA)-(E(CD))(BA)+(E(DC))(BA) \\
&= (3,3)-(3,3)-(21,3)+(21,3)-(3,3)+(3,3)+(21,3)-(21,3) \\
&\quad -(12,3)+(12,3)+(3,4)-(3,4)+(12,3)-(12,3)-(3,4)+(3,4)
\end{aligned}$$

(3,4)

$$\begin{aligned}
[[A,[B,C]],[D,E]] &= [A(BC)-A(CB)-(BC)A+(CB)A,DE-ED] \\
&= (A(BC))(DE)-(A(CB))(DE)-((BC)A)(DE)+((CB)A)(DE)-(A(BC))(ED)+(A(CB))(ED)+((BC)A)(ED)-((CB)A)(ED) \\
&\quad -(DE)(A(BC))+(DE)(A(CB))+(DE)((BC)A)-(DE)((CB)A)+(ED)(A(BC))-(ED)(A(CB))-(ED)((BC)A)+(ED)((CB)A) \\
&= (3,4)-(3,4)-(2,1)+(2,1)-(3,4)+(3,4)+(2,1)-(2,1) \\
&\quad (21,3)-(21,3)+(3,3)-(3,3)+(21,3)-(21,3)-(3,3)+(3,3)
\end{aligned}$$

(3,5)

$$\begin{aligned}
[[A,[B,[C,D]]],E] &= [A(B(CD))-A(B(DC))-A((CD)B)+A((DC)B)-(B(CD))A+(B(DC))A+((CD)B)A-((DC)B)A,E] \\
&= (A(B(CD)))E-(A(B(DC)))E-(A((CD)B))E+(A((DC)B))E-((B(CD))A)E+((B(DC))A)E+(((CD)B)A)E-(((DC)B)A)E \\
&\quad -E(A(B(CD)))+E(A(B(DC)))+E(A((CD)B))-E(A((DC)B))+E((B(CD))A)-E((B(DC))A)-E(((CD)B)A)+E(((DC)B)A) \\
&= (3,5)-(3,5)-(12,1)+(12,1)-(21,1)+(21,1)+(3,1)-(3,1) \\
&\quad +(3,6)-(3,6)+(12,2)-(12,2)+(21,2)-(21,2)-(3,2)+(3,2)
\end{aligned}$$

(3,6)

$$\begin{aligned}
[A,[B,[C,[D,E]]]] &= [A,B(C(DE))-B(C(ED))-B((DE)C)+B((ED)C)-(C(DE))B+(C(ED))B+((DE)C)B-((ED)C)B] \\
&= A(B(C(DE)))-A(B(C(ED)))-A(B((DE)C))+A(B((ED)C))-A((C(DE))B)+A((C(ED))B)+A(((DE)C)B)-A(((ED)C)B]) \\
&\quad -(B(C(DE)))A+(B(C(ED)))A+(B((DE)C))A-(B((ED)C))A+((C(DE))B)A-((C(ED))B)A-(((DE)C)B)A+(((ED)C)B)A \\
&= (3,6)-(3,6)-(12,2)+(12,2)-(21,2)+(21,2)+(3,2)-(3,2) \\
&\quad (3,5)-(3,5)+(12,1)-(12,1)+(21,1)-(21,1)-(3,1)+(3,1)
\end{aligned}$$

For the nested commutators considered so far, exactly 4 pairs of the following association types occur, $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$, $(12,1)$, $(12,2)$, $(21,1)$, $(21,2)$, $(21,3)$ $(2,1)$, whereas the two association types $(12,3)$ and $(2,2)$ do not occur at all.

(2,1)

$$\begin{aligned}
[[[A,B],[C,D]],E] &= [[AB-BA,CD-DC],E] \\
&= [(AB)(CD)-(AB)(DC)-(BA)(CD)+(BA)(DC)-(CD)(AB)+(CD)(BA)+(DC)(AB)-(DC)(BA),E] \\
&= E((AB)(CD))-E((AB)(DC))-E((BA)(CD))+E((BA)(DC))-E((CD)(AB))+E((CD)(BA))+E((DC)(AB))-E((DC)(BA)) \\
&\quad -((AB)(CD))E+((AB)(DC))E+((BA)(CD))E-((BA)(DC))E+((CD)(AB))E-((CD)(BA))E-((DC)(AB))E+((DC)(BA))E \\
&= (2,2)-(2,2)-(2,2)+(2,2)-(2,2)+(2,2)+(2,2)-(2,2) \\
&\quad -(2,1)+(2,1)+(2,1)-(2,1)+(2,1)-(2,1)-(2,1)+(2,1)
\end{aligned}$$

(2,2)

$$\begin{aligned}
[A,[[B,C],[D,E]]] &= [A,[BC-CB,DE-ED]] \\
&= [A,(BC)(DE)-(BC)(ED)-(CB)(DE)+(CB)(ED)-(DE)(BC)+(DE)(CB)+(ED)(BC)-(ED)(CB)] \\
&= A((BC)(DE))-A((BC)(ED))-A((CB)(DE))+A((CB)(ED))-A((DE)(BC))+A((DE)(CB))+A((ED)(BC))-A((ED)(CB)) \\
&\quad -((BC)(DE))A+((BC)(ED))A+((CB)(DE))A-((CB)(ED))A+((DE)(BC))A-((DE)(CB))A-((ED)(BC))A+((ED)(CB))A \\
&= (2,2)-(2,2)-(2,2)+(2,2)-(2,2)+(2,2)+(2,2)-(2,2) \\
&\quad -(2,1)+(2,1)+(2,1)-(2,1)+(2,1)-(2,1)-(2,1)+(2,1)
\end{aligned}$$

(12,1)

$$\begin{aligned}
[[A,[[B,C],D]],E] &= [[A((BC)D)-A((CB)D)-A(D(BC))+A(D(CB))-((BC)D)A+((CB)D)A+(D(BC))A-(D(CB))A,E] \\
&= (A((BC)D))E-(A((CB)D))E-(A(D(BC)))E+(A(D(CB)))E-(((BC)D)A)E+(((CB)D)A)E+((D(BC))A)E-((D(CB))A)E \\
&\quad -E(A((BC)D))+E(A((CB)D))+E(A(D(BC)))-E(A(D(CB)))+E(((BC)D)A)-E(((CB)D)A)-E((D(BC))A)+E((D(CB))A) \\
&= (12,1)-(12,1)-(3,4)+(3,4)-(3,1)+(3,1)+(21,1)-(21,1) \\
&\quad -(12,2)+(12,2)+(3,6)-(3,6)+(3,2)-(3,2)-(21,2)+(21,2)
\end{aligned}$$

(12,2)

$$\begin{aligned}
[A,[B,[[C,D],E]]] &= [A,B((CD)E)-B((DC)E)-B(E(CD))+B(E(DC))-((CD)E)B+((DC)E)B+(E(CD))B-(E(DC))B] \\
&= A(B((CD)E))-A(B((DC)E))-A(B(E(CD)))+A(B(E(DC)))-A(((CD)E)B)+A(((DC)E)B)+A((E(CD))B)-A((E(DC))B) \\
&\quad -(B((CD)E))A+(B((DC)E))A+(B(E(CD)))A-(B(E(DC)))A+(((CD)E)B)A-(((DC)E)B)A-((E(CD))B)A+((E(DC))B)A \\
&= (12,2)-(12,2)-(3,6)+(3,6)-(3,2)+(3,2)+(21,2)-(21,2) \\
&\quad -(12,1)+(12,1)+(3,5)-(3,5)+(3,1)-(3,1)-(21,1)+(21,1)
\end{aligned}$$

(12,3)

$$\begin{aligned}
[[[A,B],C],[D,E]] &= [(AB)C-(BA)C-C(AB)+C(BA),DE-ED] \\
&= ((AB)C)(DE)-((BA)C)(DE)-(C(AB))(DE)+(C(BA))(DE)-((AB)C)(ED)+((BA)C)(ED)+(C(AB))(ED)-(C(BA))(ED) \\
&\quad -(DE)((AB)C)+(DE)((BA)C)+(DE)(C(AB))-(DE)(C(BA))+(ED)((AB)C)-(ED)((BA)C)-(ED)(C(AB))+(ED)(C(BA)) \\
&= (12,3)-(12,3)-(3,4)+(3,4)-(12,3)+(12,3)+(3,4)-(3,4) \\
&\quad -(3,3)+(3,3)+(21,3)-(21,3)+(3,3)-(3,3)-(21,3)+(21,3)
\end{aligned}$$

For this cyclic order two pairs of each of the following association types occur:
$(12,1)$, $(12,2)$, $(12,3)$, $(21,1)$, $(21,2)$, $(21,3)$, $(3,1)$ $(3,2)$, $(3,3)$, $(3,4)$, $(3,5)$, $(3,6)$.
The association types $(2,1)$ and $(2,2)$ do not appear at all.

(21,1)

$$\begin{aligned}
[[[A,[B,C]],D],E] &= [(A(BC))D-(A(CB))D-((BC)A)D+((CB)A)D-D(A(BC))+D(A(CB))+D((BC)A)-D((CB)A),E] \\
&= ((A(BC))D)E-((A(CB))D)E-(((BC)A)D)E+(((CB)A)D)E-(D(A(BC)))E+(D(A(CB)))E+(D((BC)A))E-(D((CB)A))E \\
&\quad -E((A(BC))D)+E((A(CB))D)+E(((BC)A)D)-E(((CB)A)D)+E(D(A(BC)))-E(D(A(CB)))-E(D((BC)A))+E(D((CB)A)) \\
&= (21,1)-(21,1)-(3,1)+(3,1)-(3,5)+(3,5)+(12,1)-(12,1) \\
&\quad -(21,2)+(21,2)+(3,2)-(3,2)+(3,6)-(3,6)-(12,2)+(12,2)
\end{aligned}$$

(21,2)

$$\begin{aligned}
[A,[[B,[C,D]],E]] &= [A,(B(CD))E-(B(DC))E-((CD)B)E+((DC)B)E-E(B(CD))+E(B(DC))+E((CD)B)-E((DC)B)] \\
&= A((B(CD))E)-A((B(DC))E)-A(((CD)B)E)+A(((DC)B)E)-A(E(B(CD)))+A(E(B(DC)))+A(E((CD)B))-A(E((DC)B)) \\
&\quad -((B(CD))E)A+((B(DC))E)A+(((CD)B)E)A-(((DC)B)E)A+(E(B(CD)))A-(E(B(DC)))A-(E((CD)B))A+(E((DC)B))A \\
&= (21,2)-(21,2)-(3,2)+(3,2)-(3,6)+(3,6)+(12,2)-(12,2) \\
&\quad -(21,1)+(21,1)+(3,1)-(3,1)+(3,5)-(3,5)-(12,1)+(12,1)
\end{aligned}$$

(21,3)

$$\begin{aligned}
[[A,B],[C,[D,E]]] &= [AB-BA,C(DE)-C(ED)-(DE)C+(ED)C] \\
&= (AB)(C(DE))-(AB)(C(ED))-(AB)((DE)C)+(AB)((ED)C)-(BA)(C(DE))+(BA)(C(ED))+(BA)((DE)C)-(BA)((ED)C) \\
&\quad -(C(DE))(AB)+(C(ED))(AB)+((DE)C)(AB)-((ED)C)(AB)+(C(DE))(BA)-(C(ED))(BA)-((DE)C)(BA)+((ED)C)(BA) \\
&= (21,3)-(21,3)-(3,3)+(3,3)-(21,3)+(21,3)+(3,3)-(3,3) \\
&\quad -(3,4)+(3,4)+(12,3)-(12,3)+(3,4)-(3,4)-(12,3)+(12,3)
\end{aligned}$$

The association types occuring for this cyclic order are identical with the ones of the preceding cyclic order. This can be understood by noting that the two orders are reverse to one another.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Composition Algebra

A **Composition Algebra** (or *normed algebra*) is an algebra with a **multiplicative norm**.

Theorems

- Every composition algebra over a field (of characteristic not equal to 2) can be obtained by repeated application of the **Cayley-Dickson construction**.
- As composition algebras are normed algebras the **Hurwitz Theorem** applies.
- Over any field there is (up to **isomorphism**) exactly one *Split Composition Algebra* of dimension 2, 4 and 8.

A unital composition algebra is called a **Hurwitz Algebra**.

Furthermore, all triple composition algebras have been determined, up to **isotopy**, by McCrimmon.

Papers:

- Composition Algebras and their Automorphisms - N. Jacobson local pct. 136

Google books:

- Octonions, Jordan Algebras, and Exceptional Groups - T. A. Springer, F. D. Veldkamp bct. 120

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Comtrans Algebra

Comtrans Algebras are **ternary algebras** and are due to Jonathan D. H. Smith.

Their introduction (around 1988) sprang from attempts to finding an algebraic construction similar to local **Akivis algebras** for a **three-web** in the **tangent bundle** of a coordinate **n-ary loop** of a **(n+1)-web**.

The role played by comtrans algebras is analogous to the one played by **Lie algebras** of **Lie groups**. Furthermore they are analogues of **Mal'cev** and Akivis algebras.

Per definitionem, a comtrans algebra satisfies the **left alternative identity**

$$[\mathbf{A}, \mathbf{A}, \mathbf{B}] = 0$$

and consists of two ternary analogues of the binary commutator (basic trilinear operations), a **Commutator** $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$ and a **Translator** $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$, the latter satisfying the **Jacobi identity**

$$\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle + \langle \mathbf{B}, \mathbf{C}, \mathbf{A} \rangle + \langle \mathbf{C}, \mathbf{A}, \mathbf{B} \rangle = 0$$

such that together the commutator and translator obey the so called **Comtrans Identity**

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$$

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Degree 4 Association Type Expansions

In the following we assume **power-associativity** up to fourth order, otherwise the relevant monomials in the Taylor-series expansions would not be defined in a unique way.

Type 1

$$
\begin{aligned}
((\mathbf{AB})\mathbf{C})\mathbf{D}) = & \left(\left(\left(1+\mathbf{a}+\frac{1}{2}\mathbf{a}^2+\frac{1}{6}\mathbf{a}^3+\frac{1}{24}\mathbf{a}^4+\dots\right)\left(1+\mathbf{b}+\frac{1}{2}\mathbf{b}^2+\frac{1}{6}\mathbf{b}^3+\frac{1}{24}\mathbf{b}^4+\dots\right)\right)\left(1+\mathbf{c}+\frac{1}{2}\mathbf{c}^2+\frac{1}{6}\mathbf{c}^3+\frac{1}{24}c^4+\dots\right)\right) \\
& \left(1+\mathbf{d}+\frac{1}{2}\mathbf{d}^2+\frac{1}{6}\mathbf{d}^3+\frac{1}{24}\mathbf{d}^4+\dots\right) \\
= & \ 1+ \\
& \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+ \\
& \mathbf{ab}+\mathbf{ac}+\mathbf{ad}+\mathbf{bc}+\mathbf{bd}+\mathbf{cd}+\frac{1}{2}(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+\mathbf{d}^2)+ \\
& \frac{1}{2}(\mathbf{a}^2\mathbf{b}+\mathbf{a}^2\mathbf{c}+\mathbf{a}^2\mathbf{d}+\mathbf{b}^2\mathbf{c}+\mathbf{b}^2\mathbf{d}+\mathbf{c}^2\mathbf{d}+\mathbf{ab}^2+\mathbf{ac}^2+\mathbf{ad}^2+\mathbf{bc}^2+\mathbf{bd}^2+\mathbf{cd}^2)+ \\
& \frac{1}{6}(\mathbf{a}^3+\mathbf{b}^3+\mathbf{c}^3+\mathbf{d}^3)+(\mathbf{ab})\mathbf{c}+(\mathbf{ab})\mathbf{d}+(\mathbf{ac})\mathbf{d}+(\mathbf{bc})\mathbf{d}+ \\
& ((\mathbf{ab})\mathbf{c})\mathbf{d}+\frac{1}{2}((\mathbf{a}^2\mathbf{b})\mathbf{c}+(\mathbf{a}^2\mathbf{b})\mathbf{d}+(\mathbf{a}^2\mathbf{c})\mathbf{d}+(\mathbf{ab}^2)\mathbf{c}+(\mathbf{ab}^2)\mathbf{d}+(\mathbf{ac}^2)\mathbf{d}+(\mathbf{bc}^2)\mathbf{d}+(\mathbf{ab})\mathbf{c}^2+(\mathbf{ab})\mathbf{d}^2+(\mathbf{ac})\mathbf{d}^2+(\mathbf{bc})\mathbf{d}^2)+ \\
& \frac{1}{4}(\mathbf{a}^2\mathbf{b}^2+\mathbf{a}^2\mathbf{c}^2+\mathbf{a}^2\mathbf{d}^2+\mathbf{b}^2\mathbf{c}^2+\mathbf{b}^2\mathbf{d}^2+\mathbf{c}^2\mathbf{d}^2)+ \\
& \frac{1}{6}(\mathbf{a}^3\mathbf{b}+\mathbf{a}^3\mathbf{c}+\mathbf{a}^3\mathbf{d}+\mathbf{ab}^3+\mathbf{b}^3\mathbf{c}+\mathbf{b}^3\mathbf{d}+\mathbf{ac}^3+\mathbf{bc}^3+\mathbf{c}^3\mathbf{d}+\mathbf{ad}^3+\mathbf{bd}^3+\mathbf{cd}^3)+ \\
& \frac{1}{24}(\mathbf{a}^4+\mathbf{b}^4+\mathbf{c}^4+\mathbf{d}^4)+ \\
& \mathcal{O}(5)
\end{aligned}
$$

In the following only those terms are given, that differ among the 5 association types:

Type 2

$$
\begin{aligned}
\mathbf{A}((\mathbf{BC})\mathbf{D}) = & \dots+\mathbf{a}(\mathbf{bc})+\mathbf{a}(\mathbf{bd})+\mathbf{a}(\mathbf{cd})+(\mathbf{bc})\mathbf{d}+ \\
& \mathbf{a}((\mathbf{bc})\mathbf{d})+\frac{1}{2}(\mathbf{a}^2(\mathbf{bc})+\mathbf{a}^2(\mathbf{bd})+\mathbf{a}^2(\mathbf{cd})+\mathbf{a}(\mathbf{b}^2\mathbf{c})+\mathbf{a}(\mathbf{b}^2\mathbf{d})+ \\
& \mathbf{a}(\mathbf{c}^2\mathbf{d})+(\mathbf{bc}^2)\mathbf{d}+\mathbf{a}(\mathbf{bc}^2)+\mathbf{a}(\mathbf{bd}^2)+\mathbf{a}(\mathbf{cd}^2)+(\mathbf{bc})\mathbf{d}^2)+\dots+ \\
& \mathcal{O}(5)
\end{aligned}
$$

Type 3

$$
\begin{aligned}
(\mathbf{AB})(\mathbf{CD}) = & \dots+(\mathbf{ab})\mathbf{c}+(\mathbf{ab})\mathbf{d}+\mathbf{a}(\mathbf{cd})+\mathbf{b}(\mathbf{cd})+ \\
& (\mathbf{ab})(\mathbf{cd})+\frac{1}{2}((\mathbf{a}^2\mathbf{b})\mathbf{c}+(\mathbf{a}^2\mathbf{b})\mathbf{d}+\mathbf{a}^2(\mathbf{cd})+(\mathbf{ab}^2)\mathbf{c}+(\mathbf{ab}^2)\mathbf{d}+ \\
& \mathbf{a}(\mathbf{c}^2\mathbf{d})+\mathbf{b}(\mathbf{c}^2\mathbf{d})+(\mathbf{ab})\mathbf{c}^2+(\mathbf{ab})\mathbf{d}^2+\mathbf{a}(\mathbf{cd}^2)+\mathbf{b}(\mathbf{cd}^2)+\dots+ \\
& \mathcal{O}(5)
\end{aligned}
$$

Type 4

$$
\begin{aligned}
(\mathbf{A}(\mathbf{BC}))\mathbf{D} = & \dots+\mathbf{a}(\mathbf{bc})+(\mathbf{ab})\mathbf{d}+(\mathbf{ac})\mathbf{d}+(\mathbf{bc})\mathbf{d}+ \\
& (\mathbf{a}((\mathbf{bc}))\mathbf{d}+\frac{1}{2}(\mathbf{a}^2(\mathbf{bc})+(\mathbf{a}^2\mathbf{b})\mathbf{d}+(\mathbf{a}^2\mathbf{c})\mathbf{d}+\mathbf{a}(\mathbf{b}^2\mathbf{c})+(\mathbf{ab}^2)\mathbf{d}+ \\
& (\mathbf{ac}^2)\mathbf{d}+(\mathbf{bc}^2)\mathbf{d}+\mathbf{a}(\mathbf{bc}^2)+(\mathbf{ab})\mathbf{d}^2+(\mathbf{ac})\mathbf{d}^2+(\mathbf{bc})\mathbf{d}^2)+\dots+ \\
& \mathcal{O}(5)
\end{aligned}
$$

Type 5

$$
\begin{aligned}
\mathbf{A}(\mathbf{B}(\mathbf{CD})) = & \dots+\mathbf{a}(\mathbf{bc})+\mathbf{a}(\mathbf{bd})+\mathbf{a}(\mathbf{cd})+\mathbf{b}(\mathbf{cd})+ \\
& \mathbf{a}(\mathbf{b}(\mathbf{cd}))+\frac{1}{2}(\mathbf{a}^2(\mathbf{bc})+\mathbf{a}^2(\mathbf{bd})+\mathbf{a}^2(\mathbf{cd})+\mathbf{a}(\mathbf{b}^2\mathbf{c})+\mathbf{a}(\mathbf{b}^2\mathbf{d})+ \\
& \mathbf{a}(\mathbf{c}^2\mathbf{d})+\mathbf{b}(\mathbf{c}^2\mathbf{d})+\mathbf{a}(\mathbf{bc}^2)+\mathbf{a}(\mathbf{bd}^2)+\mathbf{a}(\mathbf{cd}^2)+\mathbf{b}(\mathbf{cd}^2)+\dots+ \\
& \mathcal{O}(5)
\end{aligned}
$$

See also: **Degree 5 association type expansions**.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Degree 5 Association Type Expansions

In the following we assume **power-associativity** up to fourth order, otherwise the relevant monomials in the Taylor-series expansions would not be defined in a unique way.

Type (3,1)

$$
\begin{aligned}
(((\mathbf{AB})\mathbf{C})\mathbf{D})\mathbf{E} \;=\;& \left(\left(\left(\left(1+\mathbf{a}+\frac{1}{2}\,\mathbf{a}^2+\frac{1}{6}\,\mathbf{a}^3+\frac{1}{24}\,\mathbf{a}^4+\ldots\right)\left(1+\mathbf{b}+\frac{1}{2}\,\mathbf{b}^2+\frac{1}{6}\,\mathbf{b}^3+\frac{1}{24}\,\mathbf{b}^4+\ldots\right)\right)\left(1+\mathbf{c}+\frac{1}{2}\,\mathbf{c}^2+\frac{1}{6}\,\mathbf{c}^3+\frac{1}{24}\,c^4+\ldots\right)\right)\right.\\
&\left.\left(1+\mathbf{d}+\frac{1}{2}\,\mathbf{d}^2+\frac{1}{6}\,\mathbf{d}^3+\frac{1}{24}\,\mathbf{d}^4+\ldots\right)\right)\left(1+\mathbf{e}+\frac{1}{2}\,\mathbf{e}^2+\frac{1}{6}\,\mathbf{e}^3+\frac{1}{24}\,\mathbf{e}^4+\ldots\right)\\
=\;& 1+\\
& \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}\\
& \mathbf{ab}+\mathbf{ac}+\mathbf{ad}+\mathbf{ae}+\mathbf{bc}+\mathbf{bd}+\mathbf{be}+\mathbf{cd}+\mathbf{ce}+\mathbf{de}+\frac{1}{2}\,(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+\mathbf{d}^2+\mathbf{e}^2)+\\
& \frac{1}{2}\,(\mathbf{a}^2\mathbf{b}+\mathbf{a}^2\mathbf{c}+\mathbf{a}^2\mathbf{d}+\mathbf{a}^2\mathbf{e}+\mathbf{b}^2\mathbf{c}+\mathbf{b}^2\mathbf{d}+\mathbf{b}^2\mathbf{e}+\mathbf{c}^2\mathbf{d}+\mathbf{c}^2\mathbf{e}+\mathbf{d}^2\mathbf{e}\\
& \mathbf{ab}^2+\mathbf{ac}^2+\mathbf{ad}^2+\mathbf{ae}^2+\mathbf{bc}^2+\mathbf{bd}^2+\mathbf{be}^2+\mathbf{cd}^2+\mathbf{ce}^2+\mathbf{de}^2)+\\
& \frac{1}{6}\,(\mathbf{a}^3+\mathbf{b}^3+\mathbf{c}^3+\mathbf{d}^3+\mathbf{e}^3)+\\
& (\mathbf{ab})\mathbf{c}+(\mathbf{ab})\mathbf{d}+(\mathbf{ab})\mathbf{e}+(\mathbf{ac})\mathbf{d}+(\mathbf{ac})\mathbf{e}+(\mathbf{ad})\mathbf{e}+(\mathbf{bc})\mathbf{d}+(\mathbf{bc})\mathbf{e}+(\mathbf{bd})\mathbf{e}+(\mathbf{cd})\mathbf{e}+\\
& ((\mathbf{ab})\mathbf{c})\mathbf{d}+((\mathbf{ab})\mathbf{c})\mathbf{e}+((\mathbf{ab})\mathbf{d})\mathbf{e}+((\mathbf{ac})\mathbf{d})\mathbf{e}+((\mathbf{bc})\mathbf{d})\mathbf{e}+\\
& \frac{1}{2}\,((\mathbf{a}^2\mathbf{b})\mathbf{c}+(\mathbf{a}^2\mathbf{b})\mathbf{d}+(\mathbf{a}^2\mathbf{b})\mathbf{e}+(\mathbf{a}^2\mathbf{c})\mathbf{d}+(\mathbf{a}^2\mathbf{c})\mathbf{e}+(\mathbf{a}^2\mathbf{d})\mathbf{e}+(\mathbf{ab}^2)\mathbf{c}+(\mathbf{ab}^2)\mathbf{d}+(\mathbf{ab}^2)\mathbf{e}+\\
& (\mathbf{ac}^2)\mathbf{d}+(\mathbf{ac}^2)\mathbf{e}+(\mathbf{bc}^2)\mathbf{d}+(\mathbf{bc}^2)\mathbf{e}+(\mathbf{ab})\mathbf{c}^2+(\mathbf{ad}^2)\mathbf{e}+(\mathbf{bd}^2)\mathbf{e}+(\mathbf{cd}^2)\mathbf{e}+(\mathbf{ab})\mathbf{d}^2+(\mathbf{ac})\mathbf{d}^2+(\mathbf{bc})\mathbf{d}^2\\
& (\mathbf{ab})\mathbf{e}^2+(\mathbf{ac})\mathbf{e}^2+(\mathbf{ad})\mathbf{e}^2+(\mathbf{bc})\mathbf{e}^2+(\mathbf{bd})\mathbf{e}^2+(\mathbf{cd})\mathbf{e}^2)+\\
& \frac{1}{4}\,(\mathbf{a}^2\mathbf{b}^2+\mathbf{a}^2\mathbf{c}^2+\mathbf{a}^2\mathbf{d}^2+\mathbf{a}^2\mathbf{e}^2+\mathbf{b}^2\mathbf{c}^2+\mathbf{b}^2\mathbf{d}^2+\mathbf{b}^2\mathbf{e}^2+\mathbf{c}^2\mathbf{d}^2+\mathbf{c}^2\mathbf{e}^2+\mathbf{d}^2\mathbf{e}^2)+\\
& \frac{1}{6}\,(\mathbf{a}^3\mathbf{b}+\mathbf{a}^3\mathbf{c}+\mathbf{a}^3\mathbf{d}+\mathbf{a}^3\mathbf{e}+\mathbf{ab}^3+\mathbf{b}^3\mathbf{c}+\mathbf{b}^3\mathbf{d}+\mathbf{b}^3\mathbf{e}+\mathbf{ac}^3+\mathbf{bc}^3+\mathbf{c}^3\mathbf{d}+\mathbf{c}^3\mathbf{e}+\\
& \mathbf{ad}^3+\mathbf{bd}^3+\mathbf{cd}^3+\mathbf{d}^3\mathbf{e}+\mathbf{ae}^3+\mathbf{be}^3+\mathbf{ce}^3+\mathbf{de}^3+\mathbf{de}^3)+\\
& \frac{1}{24}\,(\mathbf{a}^4+\mathbf{b}^4+\mathbf{c}^4+\mathbf{d}^4+\mathbf{e}^4)+\\
& \mathcal{O}(5)
\end{aligned}
$$

The first and the second order are the same for all **association types**. The same applies for all terms of order three that are not totally inhomogeneous. In fourth order, terms containing two times a squared element, a cubic or a hypercubic element are the same. Henceforward all these terms will be omitted.

Type (3,2)

$$
\begin{aligned}
\mathbf{A}(((\mathbf{BC})\mathbf{D})\mathbf{E}) \;=\;& \left(1+\mathbf{a}+\frac{1}{2}\,\mathbf{a}^2+\frac{1}{6}\,\mathbf{a}^3+\frac{1}{24}\,\mathbf{a}^4+\ldots\right)\left(\left(\left(\left(1+\mathbf{b}+\frac{1}{2}\,\mathbf{b}^2+\frac{1}{6}\,\mathbf{b}^3+\frac{1}{24}\,\mathbf{b}^4+\ldots\right)\left(1+\mathbf{c}+\frac{1}{2}\,\mathbf{c}^2+\frac{1}{6}\,\mathbf{c}^3+\frac{1}{24}\,c^4+\ldots\right)\right)\right.\right.\\
&\left.\left.\left(1+\mathbf{d}+\frac{1}{2}\,\mathbf{d}^2+\frac{1}{6}\,\mathbf{d}^3+\frac{1}{24}\,\mathbf{d}^4+\ldots\right)\right)\left(1+\mathbf{e}+\frac{1}{2}\,\mathbf{e}^2+\frac{1}{6}\,\mathbf{e}^3+\frac{1}{24}\,\mathbf{e}^4+\ldots\right)\right)\\
=\;& \ldots+\mathbf{a}(\mathbf{bc})+\mathbf{a}(\mathbf{bd})+\mathbf{a}(\mathbf{be})+\mathbf{a}(\mathbf{cd})+\mathbf{a}(\mathbf{ce})+\mathbf{a}(\mathbf{de})+(\mathbf{bc})\mathbf{d}+(\mathbf{bc})\mathbf{e}+(\mathbf{bd})\mathbf{e}+(\mathbf{cd})\mathbf{e}+\\
& \mathbf{a}((\mathbf{bc})\mathbf{d})+\mathbf{a}((\mathbf{bc})\mathbf{e})+\mathbf{a}((\mathbf{bd})\mathbf{e})+\mathbf{a}((\mathbf{cd})\mathbf{e})+((\mathbf{bc})\mathbf{d})\mathbf{e}+\\
& \frac{1}{2}\,(\mathbf{a}^2(\mathbf{bc})+\mathbf{a}^2(\mathbf{bd})+\mathbf{a}^2(\mathbf{be})+\mathbf{a}^2(\mathbf{cd})+\mathbf{a}^2(\mathbf{ce})+\mathbf{a}^2(\mathbf{de})+\mathbf{a}(\mathbf{b}^2\mathbf{c})+\mathbf{a}(\mathbf{b}^2\mathbf{d})+\mathbf{a}(\mathbf{b}^2\mathbf{e})+\\
& \mathbf{a}(\mathbf{c}^2\mathbf{d})+\mathbf{a}(\mathbf{c}^2\mathbf{e})+(\mathbf{bc}^2)\mathbf{d}+(\mathbf{bc}^2)\mathbf{e}+\mathbf{a}(\mathbf{bc}^2)+\mathbf{a}(\mathbf{d}^2\mathbf{e})+(\mathbf{bd}^2)\mathbf{e}+(\mathbf{cd}^2)\mathbf{e}+\mathbf{a}(\mathbf{bd}^2)+\mathbf{a}(\mathbf{cd}^2)+(\mathbf{bc})\mathbf{d}^2+\\
& \mathbf{a}(\mathbf{be}^2)+\mathbf{a}(\mathbf{ce}^2)+\mathbf{a}(\mathbf{de}^2)+(\mathbf{bc})\mathbf{e}^2+(\mathbf{bd})\mathbf{e}^2+(\mathbf{cd})\mathbf{e}^2)+\ldots\\
& \mathcal{O}(5)
\end{aligned}
$$

Type (3,3)

$$
\begin{aligned}
(\mathbf{AB})((\mathbf{CD})\mathbf{E}) \;=\;& \left(\left(1+\mathbf{a}+\frac{1}{2}\,\mathbf{a}^2+\frac{1}{6}\,\mathbf{a}^3+\frac{1}{24}\,\mathbf{a}^4+\ldots\right)\left(1+\mathbf{b}+\frac{1}{2}\,\mathbf{b}^2+\frac{1}{6}\,\mathbf{b}^3+\frac{1}{24}\,\mathbf{b}^4+\ldots\right)\right)\left(\left(\left(1+\mathbf{c}+\frac{1}{2}\,\mathbf{c}^2+\frac{1}{6}\,\mathbf{c}^3+\frac{1}{24}\,c^4+\ldots\right)\right.\right.\\
&\left.\left.\left(1+\mathbf{d}+\frac{1}{2}\,\mathbf{d}^2+\frac{1}{6}\,\mathbf{d}^3+\frac{1}{24}\,\mathbf{d}^4+\ldots\right)\right)\left(1+\mathbf{e}+\frac{1}{2}\,\mathbf{e}^2+\frac{1}{6}\,\mathbf{e}^3+\frac{1}{24}\,\mathbf{e}^4+\ldots\right)\right)\\
=\;& \ldots+(\mathbf{ab})\mathbf{c}+(\mathbf{ab})\mathbf{d}+(\mathbf{ab})\mathbf{e}+\mathbf{a}(\mathbf{cd})+\mathbf{a}(\mathbf{ce})+\mathbf{a}(\mathbf{de})+\mathbf{b}(\mathbf{cd})+\mathbf{b}(\mathbf{ce})+\mathbf{b}(\mathbf{de})+(\mathbf{cd})\mathbf{e}+\\
& (\mathbf{ab})(\mathbf{cd})+(\mathbf{ab})(\mathbf{ce})+(\mathbf{ab})(\mathbf{de})+\mathbf{a}((\mathbf{cd})\mathbf{d})+\mathbf{b}((\mathbf{cd})\mathbf{e})+\\
& \frac{1}{2}\,((\mathbf{a}^2\mathbf{b})\mathbf{c}+(\mathbf{a}^2\mathbf{b})\mathbf{d}+(\mathbf{a}^2\mathbf{b})\mathbf{e}+\mathbf{a}^2(\mathbf{cd})+\mathbf{a}^2(\mathbf{ce})+\mathbf{a}^2(\mathbf{de})+(\mathbf{ab}^2)\mathbf{c}+(\mathbf{ab}^2)\mathbf{d}+(\mathbf{ab}^2)\mathbf{e}+\\
& \mathbf{a}(\mathbf{c}^2\mathbf{d})+\mathbf{a}(\mathbf{c}^2\mathbf{e})+\mathbf{b}(\mathbf{c}^2\mathbf{d})+\mathbf{b}(\mathbf{c}^2\mathbf{e})+(\mathbf{ab})\mathbf{c}^2+\mathbf{a}(\mathbf{d}^2\mathbf{e})+\mathbf{b}(\mathbf{d}^2\mathbf{e})+(\mathbf{cd}^2)\mathbf{e}+(\mathbf{ab})\mathbf{d}^2)+\mathbf{a}(\mathbf{cd}^2)+\mathbf{b}(\mathbf{cd}^2)+\\
& (\mathbf{ab})\mathbf{e}^2+\mathbf{a}(\mathbf{ce}^2)+\mathbf{a}(\mathbf{de}^2)+\mathbf{b}(\mathbf{ce}^2)+\mathbf{b}(\mathbf{de}^2)+(\mathbf{cd})\mathbf{e}^2)+\ldots\\
& \mathcal{O}(5)
\end{aligned}
$$

Type (3,4)

TODO to be continued ..

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Diassociativity

A **loop** $\mathcal{L}$ is called **Diassociative** if any pair of elements of $\mathcal{L}$ generates a **group** in $\mathcal{L}$.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Division Algebra

An algebra $\mathcal{A}$ is called a **Division algebra** if it possesses no **zero-divisors**. I.e. for any element $\mathbf{A} \in \mathcal{A}$ and any non-zero element $\mathbf{B} \in \mathcal{A}$ there exists exactly one element $\mathbf{X} \in \mathcal{A}$ and $\mathbf{Y} \in \mathcal{A}$ respectively, such that $\mathbf{A} = \mathbf{BX}$ and $\mathbf{A} = \mathbf{YB}$.

Division algebras are also referred to as **compact** algebras.

Theorem (M. Kervaire, J. Milnor)
Any finite-dimensional real division algebra must be of dimension 1, 2, 4, or 8.

However over the *p-adic numbers* there are an infinite number of division algebras.

An example of a division algebra of order 16 over the rational numbers is described in [1] and [2]. It is based on a modified **Cayley-Dickson doubling process**, yet it doesn't yield **alternative algebras** if applied to the *complex numbers* or the real **quaternion algebra**.

Papers:

- [1] On a Construction for Division Algebras of Order 16 (1945) - R. D. Schafer local pct. 2
- [2] Equivalence in a Class of Division Algebras of Order 16 (1946) - R. D. Schafer local pct. 0

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Dual Number

Dual Numbers are a variant of *complex numbers*, having a basis $\{\mathbf{e}, \mathbf{e}_1\}$ with a **nilpotent** "imaginary unit", i.e. $\mathbf{e}_1^2 = 0$.

Dual numbers constitute one of the simplest non-trivial examples of a **superspace**. The direction along $\mathbf{e}_1$ is termed the "fermionic" direction, and the real component is termed the "bosonic" direction.

Links:

- WIKIPEDIA - Dual Number

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Extra Loop

Papers:

- The Structure of Extra Loops (2004) - M. K. Kinyon, K. Kunen local pct. 29

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Factor

A **Factor** is a **Von Neumann algebra** with a **center** only consisting of multiples of the identity operator.

Murray & von Neumann (1936) showed that every factor is of type I, II or III.

Von Neumann (1949) showed that every von Neumann algebra on a **separable Hilbert space** is **isomorphic** to a direct integral of factors. This decomposition is essentially unique. Thus, the problem of classifying isomorphism classes of von Neumann algebras on separable Hilbert spaces can be reduced to that of classifying isomorphism classes of factors.

Type I_n: $d: \{0, 1, \ldots, n\}$ where n is a natural number
Physical example: Non-relativistic finite dimensional **quantum mechanics** ($n \times n$ complex matrices).
If $n = 1$ the Von Neumann algebra is commutative.

Type I_∞: $d: \{0, 1, \ldots, \infty\}$
Physical example: Free boson **Fock space**.

Type II_1: $d: [0, 1]$
Physical examples:
Free fermion Fock space.
Infinite temperature maximally **chaotic** *KMS state*.

Type II_∞: $d: [0, \infty)$
Physical example: The **tensor product** of the free boson and fermion Fock spaces.

Type III: $d: \{0, \infty\}$ (two-element set)
Physical examples:
Local algebras in **AQFT**.
KMS states with finite non-zero temperature in AQSM.

Papers:

- On Factor Representations and the C*-Algebra of Canonical Commutation Relations (1972) - J. Slawny local pct. 140

Links:

- WIKIPEDIA - Von Neumann Algebra
- WIKIPEDIA - Typklassifikation Von-Neumann-Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

First Bianchi Identity

The **First Bianchi Identity** (a.k.a. **"Algebraic Bianchi Identity"** or **Ricci Identity**) establishes the relationship between the **torsion tensor** $T^\sigma_{\mu\nu}$ and the **nonassociativity tensor** $A_{\mu\nu\rho}{}^\sigma$.
It therefore describes the interplay between the second order and the third order of an algebra in its **tangent space**.
In its most general form the identity reads:

$$J_{\mu\nu\rho}{}^\sigma = 4T_{[\mu\nu}{}^\tau T_{|\tau|\rho]}{}^\sigma \equiv A_{[\mu\nu\rho]}{}^\sigma$$

with $J_{\mu\nu\rho}{}^\sigma$ the **(commutator) Jacobi tensor**.

If the Jacobi tensor is equal to zero, one is on the level of a second-order (i.e. an associative) algebra, where one finds the classical relationship between the torsion and the Riemann tensor, given by

$$0 = J_{\mu\nu\rho}{}^\sigma = R_{[\mu\nu\rho]}{}^\sigma - 2D_{[\mu}T_{\nu\rho]}{}^\sigma$$

The third order level of an algebra, represented by the nonassociativity tensor is therefore related to the second order level of an algebra in the following way

$$A_{[\mu\nu\rho]}{}^\sigma = 4T_{[\mu\nu}{}^\tau T_{|\tau|\rho]}{}^\sigma = R_{[\mu\nu\rho]}{}^\sigma - 2D_{[\mu}T_{\nu\rho]}{}^\sigma$$

$A_{\mu\nu\rho}{}^\sigma$ and $T_{\mu\nu}{}^\rho$ are fundamental tensors, corresponding to the respective order of the algebra, whereas the Riemann tensor $R_{\mu\nu\rho}{}^\sigma$ is not. (This sheds some light on the fact that Riemannian geometry is a special geometry, naturally embedded in a more general one, namely in **nonassociative geometry**).

Another way to see it is as follows:
Given a second order algebra, one has

$$D_{[\mu}T_{\nu\rho]}{}^\sigma = \frac{1}{2} R_{[\mu\nu\rho]}{}^\sigma$$

Making the transition to a third order algebra requires the addition of a correction term, hence

$$D_{[\mu}T_{\nu\rho]}{}^\sigma = \frac{1}{2} R_{[\mu\nu\rho]}{}^\sigma - \frac{1}{2} A_{[\mu\nu\rho]}{}^\sigma$$

The important thing is, that this correction term is a fundamentally new object brought in by the higher order algebra (i.e. its information is only coded in the *structure constants* of the third order algebra). Contrary to the Riemann tensor, which can be expressed in terms of the torsion tensor on the level of a second order algebra, the non-associativity tensor is an independent object (and hence another fundamental tensor). On algebraic grounds this means that in the same way as the torsion tensor corresponds with a fundamental property of the underlying algebra, namely with noncommutativity, the nonassociativity tensor corresponds with nonassociativity. A second order algebra cannot possess the latter property, similarly to a first order algebra, which cannot possess the property of noncommutativity.

The first Bianchi identity can equivalently be expressed as follows

$$\begin{aligned}\mathbf{A}(\boldsymbol{\partial}_{[\mu}, \boldsymbol{\partial}_\nu, \boldsymbol{\partial}_{\rho]}) &= 4\mathbf{T}(\mathbf{T}(\boldsymbol{\partial}_{[\mu}, \boldsymbol{\partial}_\nu), \boldsymbol{\partial}_{\rho]}) \\ &= \mathbf{R}(\boldsymbol{\partial}_{[\mu}, \boldsymbol{\partial}_\nu)\boldsymbol{\partial}_{\rho]} - 2\boldsymbol{\partial}_{[\mu}\mathbf{T}(\boldsymbol{\partial}_\nu, \boldsymbol{\partial}_{\rho]})\end{aligned}$$

which shows that it satisfies the **Akivis relation**.

Proof:
Φ will denote a scalar function in the following.

In terms of torsion we have

$$\begin{aligned}[[\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu], \boldsymbol{\partial}_\rho]\Phi &= [\mathbf{T}_{\mu\nu}, \boldsymbol{\partial}_\rho]\Phi \\ &= [\mathbf{T}(\mathbf{e}_\mu, \mathbf{e}_\nu), \boldsymbol{\partial}_\rho]\Phi \\ &= \mathbf{T}(\mathbf{T}(\mathbf{e}_\mu, \mathbf{e}_\nu), \mathbf{e}_\rho)\Phi\end{aligned}$$

and in terms of non-associativity

$$\begin{aligned}[[\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu], \boldsymbol{\partial}_\rho]\Phi &= [\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu]\boldsymbol{\partial}_\rho\Phi - \boldsymbol{\partial}_\rho[\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu]\Phi \\ &= [\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu]\mathbf{e}_\rho\Phi - \boldsymbol{\partial}_\rho\mathbf{T}(\mathbf{e}_\mu, \mathbf{e}_\nu)\Phi \\ &= \mathbf{R}(\mathbf{e}_\mu, \mathbf{e}_\nu)\mathbf{e}_\rho\Phi - \boldsymbol{\partial}_\rho\mathbf{T}(\mathbf{e}_\mu, \mathbf{e}_\nu)\Phi\end{aligned}$$

Cycling the indices appropriately, adding the resulting terms and taking care of the factors leads to the identities mentioned.

...

The Akivis identity is just a repackaging of the algebraic elements involved. Hence the same must be true for the first Bianchi identity, which can be seen in the proof above. The only **symmetry of the Riemann tensor** that is implicit is the antisymmetry due to the commutator involved. The resulting Bianchi identity coincides with the one obtained when assuming a most general *connection*. This implies that such a general connection allows for a manifold that is based on a nonassociative (and/or noncommutative) algebra (i.e. **loop manifolds**). Its description consequently is based on *nonassociative differential geometry*. (This seems to bear little attention in literature when mentioning the Bianchi identity in this general form).

Examples

Lie algebra

As a Lie Algebra is associative, the anassociativity tensor must vanish, which is equivalent to saying that the Jacobi identity must be satisfied, i.e. the Jacobian must be zero. One is left with the condition

$$R_{[\mu\nu\rho]}{}^\sigma - 2D_{[\mu}T_{\nu\rho]}{}^\sigma = 0$$

In other words, the first Bianchi identity imposes the constraint that the third order of the **tangent space** does not exist.
As a consequence, knowing the commutation relations of a Lie algebra is sufficient to characterise it completely. (This is the "ususal business" in physics).
On the other hand, characterizing higher order algebras requires not just writing down commutation relations but also **"Jacobiator** relations" etc.

Riemann space

In this case **torsion** and its covariant derivatives are zero and the Bianchi identities simplify to

$$J_{\mu\nu\rho}{}^\sigma = A_{[\mu\nu\rho]}{}^\sigma = R^\sigma_{[\mu\nu\rho]} = R^\sigma_{\mu\nu\rho} + R^\sigma_{\nu\rho\mu} + R^\sigma_{\rho\mu\nu} = 0$$

This implies that $A_{\mu\nu\rho}{}^\sigma$ which is a measure of the degree of nonassociativity of the algebra describing the manifold also vanishes.

Second order analogue

As the first Bianchi identity corresponds with the third order of the tangent space and the *second Bianchi identity* with the fourth order, one might wonder what corresponds with the second order.
It is given by the relationship of the **vielbeins** and the torsion tensor (which hence represents the relationship of the first order and the second order of the tangent space).

$$h^\sigma_{[\mu}\partial_\sigma h^\rho_{\nu]} = T^\rho_{\mu\nu}$$

This can equivalently expressed as

$$[\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu] = T_{\mu\nu}{}^\alpha\boldsymbol{\partial}_\alpha$$

As one is dealing with a tensor relationship, one can transform this identity to an orthonormal frame:

$$[\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu] \to [\boldsymbol{\partial}_a, \boldsymbol{\partial}_b]$$

The torsion tensor then is to be replaced by the structure constants and one gets

$$[\boldsymbol{\partial}_a, \boldsymbol{\partial}_b] = C^C_{ab}\boldsymbol{\partial}_C$$

The *structure constants* are determined by the algebra, i.e. one needs input from the outside how the second order of the tangent space should look like. For example, in a **Riemann space** this expression is zero, as there is no torsion in this case.
If the structure constants all vanish, one says that the **"integrability condition"** $\boldsymbol{\partial}_a\boldsymbol{\partial}_b - \boldsymbol{\partial}_b\boldsymbol{\partial}_a = 0$ is satisfied, i.e. the derivatives commute. (This is the so called Schwarz's theorem or Clairaut's theorem which therefore is to be seen as the analogue of the Jacobi identity, corresponding with the first Bianchi identity).
If they don't, one says that the coordinates are **nonholonomic**. (In **quantum mechanics** one would also speak of an "anomalous commutator").

It should be stressed, that without the input, the identity is "empty". I.e. if one resolves both sides of the equation, one gets two identical expressions (a fact that might be confusing at first sight). The crux is, that if one specifies the algebra and calculates one side of the equation, the result is determined by the "inner workings of the algebraic product".
E.g. for two complex numbers $\mathbf{A}$, $\mathbf{B}$ on has

$$[\mathbf{A}, \mathbf{B}] = 0$$

whereas if they were **quaternionic** one had

$$[\mathbf{A},\mathbf{B}] \neq 0$$

Orthonormal Frames

In complete analogy with the second order of the tangent space we can proceed for the third order, i.e. for the first Bianchi identity.

Again, using the fact that the first Bianchi identity is tensorial, we can do a transformation to an orthonormal frame, i.e.

$$\sigma_{\{\mu,\nu,\rho\}}[[\boldsymbol{\partial}_\mu,\boldsymbol{\partial}_\nu],\boldsymbol{\partial}_\rho] \to \sigma_{\{a,b,c\}}[[\boldsymbol{\partial}_a,\boldsymbol{\partial}_b],\boldsymbol{\partial}_c]$$

Here the nonassociativity tensor is to be replaced by the corresponding structure constants and one gets

$$[[\boldsymbol{\partial}_a,\boldsymbol{\partial}_b],\boldsymbol{\partial}_c] = C^D_{abc}\boldsymbol{\partial}_D$$

Again input is required, as to how the manifold in terms of the third order of its tangent space should look like.
If the structure constants all vanish, it does not contribute and one is dealing with the classical Jacobi identity, characteristic of an associative algebra. (This is also referred to as an integrability condition, this time corresponding with the third order).
Else, one says that the Jacobian is "anomalous" (supposedly expressing the fact, that those who coined the word didn't understand what that means). We'll therefore call related coordinates henceforward - in analogy to the second order case - **Anomalous Coordinates**.
An example are the **octonions**, where in an orthonormal frame one has

$$[[\boldsymbol{\partial}_a,\boldsymbol{\partial}_b],\boldsymbol{\partial}_c] = C^D_{abc}\boldsymbol{\partial}_D$$

with $a,b,c = 1,2,3$ and $D = 1,\ldots,7$.
The octonionic product is defined by

$$\mathbf{E}_A\mathbf{E}_B \equiv f^C_{AB}\mathbf{E}_C$$

where $A,B,C = 1,\ldots 7$. The third order structure constants f^C_{AB} are totally antisymmetric.

To demonstrate how one gets the C^D_{abc} from the f^C_{AB}, we'll make an example,

$$[[\boldsymbol{\partial}_2,\boldsymbol{\partial}_3],\boldsymbol{\partial}_1] = C^D_{231}\boldsymbol{\partial}_D$$

Except for the unitiy element, the octonions consist of 7 basis elements, 3 vectors $\boldsymbol{\partial}_1$, $\boldsymbol{\partial}_2$, $\boldsymbol{\partial}_3$, 3 bivectors $\boldsymbol{\partial}_{12} \equiv \boldsymbol{\partial}_4$, $\boldsymbol{\partial}_{13} \equiv \boldsymbol{\partial}_5$, $\boldsymbol{\partial}_{23} \equiv \boldsymbol{\partial}_6$ and a trivector $\boldsymbol{\partial}_{123} \equiv \boldsymbol{\partial}_7$.
Thus we can rewrite our example as follows (being a bit sloppy with the signs which don't matter for the sake of this demonstration):

$$\begin{aligned}
[[\boldsymbol{\partial}_2,\boldsymbol{\partial}_3],\boldsymbol{\partial}_1] &= [\boldsymbol{\partial}_{23}-\boldsymbol{\partial}_{32},\boldsymbol{\partial}_1] \\
&= 2[\boldsymbol{\partial}_{23},\boldsymbol{\partial}_1] \\
&= 2[\boldsymbol{\partial}_6,\boldsymbol{\partial}_1] \\
&= -4\boldsymbol{\partial}_1\boldsymbol{\partial}_6 \\
&= f^7_{16}\boldsymbol{\partial}_7 \\
&= C^5_{231}\boldsymbol{\partial}_5
\end{aligned}$$

Therefore the structure constants defined by a binary product (which may be interpreted as graded or may not) suffice to specify higher orders of the tangent space, characterised by higher order products. (This is an essential motivation for introducing the concept of a **polyvector space**).

See also:

- *Second Bianchi identity*

Papers:

- Torsion and Attractors in the Kolmogorov Hydrodynamical System (1998) - A. Pasini, V. Pelino, S. Potestà local pct. 8
- Spontaneous Compactification and Nonassociativity (2009) - E. K. Loginov local pct. 2

Lectures:

- Notes on Differential Geometry (2011) - M. Visser local lrl. 9 - Excellent lectures !

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Flexible Algebra

> Most of the interesting non-associative algebras are flexible.
> - Susumu Okubo - Introduction to Octonion and other Non-associative Algebras in Physics

An **algebra** is called **Flexible** if it satisfies the condition:

$$[\mathbf{A},\mathbf{B},\mathbf{A}] = 0$$

i.e.

$$(\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA})$$

or equivalently

$$L_{\mathbf{A}}R_{\mathbf{A}}(\mathbf{B}) = R_{\mathbf{A}}L_{\mathbf{A}}(\mathbf{B})$$

with L and R denoting the **left- and right translation operators** respectively.

Flexibility implies **monoassociativity**.

An algebra $\mathcal{A}$ is flexible if and only if the identity

$$[\mathbf{D}, \mathbf{A} \circ \mathbf{B}] = [\mathbf{D}, \mathbf{A}] \circ \mathbf{B} + \mathbf{A} \circ [\mathbf{D}, \mathbf{B}]$$

holds for $\forall \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{A}$. I.e. the map $ad_{\mathbf{D}} : \mathcal{A} \to \mathcal{A}$ defined by $ad_{\mathbf{D}}(\mathbf{A}) = [\mathbf{D}, \mathbf{A}]$ is a *derivation* of the **commutative algebra** A^+.

Linearisation leads to

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = -[\mathbf{C}, \mathbf{B}, \mathbf{A}]$$

i.e. the **associator** is antisymmetric in the 1^{st} and 3^{rd} component.

Due to flexibility the **Akivis identity** simplifies to

$$[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = 2[\mathbf{A}, \mathbf{B}, \mathbf{C}] + 2[\mathbf{B}, \mathbf{C}, \mathbf{A}] + 2[\mathbf{C}, \mathbf{A}, \mathbf{B}]$$

or in terms of the **commutator Jacobian** and a *cyclic sum*

$$\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) = 2\sigma_{\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}}[\mathbf{A}, \mathbf{B}, \mathbf{C}]$$

The *Teichmüller identity* can be expressed as:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}, \mathbf{D}]] + [[\mathbf{A}, \mathbf{B}, \mathbf{C}], \mathbf{D}] = [\mathbf{A}, \mathbf{B}, [\mathbf{C}, \mathbf{D}]] - [\mathbf{A}, [\mathbf{B}, \mathbf{C}], \mathbf{D}] + [[\mathbf{A}, \mathbf{B}], \mathbf{C}, \mathbf{D}]$$

Any flexible quadratic algebra is a **noncommutative Jordan algebra**.

Further identities:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}], \mathbf{D}] = [[\mathbf{B}, \mathbf{C}], \mathbf{D}, \mathbf{A}]$$
$$\langle \mathbf{A} | \mathbf{B}\mathbf{C} \rangle = \langle \mathbf{B}^* \mathbf{A} | \mathbf{C} \rangle = \langle \mathbf{A}\mathbf{C}^* | \mathbf{B} \rangle$$
$$(\mathbf{B}\mathbf{A}^*)\mathbf{A} = \mathbf{A}^*(\mathbf{A}\mathbf{B})$$

A flexible algebra need not to be **power associative**.

Papers:

- Loops with Universal Elasticity (1994) - P. N. Syrbu local pct. 7

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Fourth Order Bol Identities Expansions

Right Bol Loop

The **right Bol identity** reads

$$((\mathbf{A}\mathbf{B})\mathbf{C})\mathbf{B} = \mathbf{A}((\mathbf{B}\mathbf{C})\mathbf{B})$$

The first three orders of the Taylor series expansion of this identity are described under **association type expansions**.
In the following we'll examine the fourth order of the expansion.

Using the results for the **degree 4 association type expansions** of order four for the **association types** 1 and 2 and setting them equal, one gets

$$((\mathbf{ab})\mathbf{c})\mathbf{b} + \frac{1}{2}\left((\mathbf{a}^2\mathbf{b})\mathbf{c} + (\mathbf{a}^2\mathbf{b})\mathbf{b} + (\mathbf{a}^2\mathbf{c})\mathbf{b} + (\mathbf{ab}^2)\mathbf{c} + (\mathbf{ab}^2)\mathbf{b} + (\mathbf{ac}^2)\mathbf{b} + (\mathbf{bc}^2)\mathbf{b} + (\mathbf{ab})\mathbf{c}^2 + (\mathbf{ab})\mathbf{b}^2 + (\mathbf{ac})\mathbf{b}^2 + (\mathbf{bc})\mathbf{b}^2\right)\right) =$$
$$\mathbf{a}((\mathbf{bc})\mathbf{b}) + \frac{1}{2}\left(\mathbf{a}^2(\mathbf{bc}) + \mathbf{a}^2(\mathbf{bb}) + \mathbf{a}^2(\mathbf{cb}) + \mathbf{a}(\mathbf{b}^2\mathbf{c}) + \mathbf{a}(\mathbf{b}^2\mathbf{b}) + \mathbf{a}(\mathbf{c}^2\mathbf{b}) + (\mathbf{bc}^2)\mathbf{b} + \mathbf{a}(\mathbf{bc}^2) + \mathbf{a}(\mathbf{bb}^2) + \mathbf{a}(\mathbf{cb}^2) + (\mathbf{bc})\mathbf{b}^2)\right)$$

and, after simplifications

$$((\mathbf{ab})\mathbf{c})\mathbf{b} + \frac{1}{2}\left((\mathbf{a}^2\mathbf{b})\mathbf{c} + (\mathbf{a}^2\mathbf{b})\mathbf{b} + (\mathbf{a}^2\mathbf{c})\mathbf{b} + (\mathbf{ab}^2)\mathbf{c} + (\mathbf{ab}^2)\mathbf{b} + (\mathbf{ac}^2)\mathbf{b} + (\mathbf{ab})\mathbf{c}^2 + (\mathbf{ab})\mathbf{b}^2 + (\mathbf{ac})\mathbf{b}^2\right)\right) =$$
$$\mathbf{a}((\mathbf{bc})\mathbf{b}) + \frac{1}{2}\left(\mathbf{a}^2(\mathbf{bc}) + \mathbf{a}^2(\mathbf{bb}) + \mathbf{a}^2(\mathbf{cb}) + \mathbf{a}(\mathbf{b}^2\mathbf{c}) + \mathbf{a}(\mathbf{b}^2\mathbf{b}) + \mathbf{a}(\mathbf{c}^2\mathbf{b}) + \mathbf{a}(\mathbf{bc}^2) + \mathbf{a}(\mathbf{bb}^2) + \mathbf{a}(\mathbf{cb}^2)\right)$$

Expressed in terms of **associators** this is

$$((\mathbf{ab})\mathbf{c})\mathbf{b} - \mathbf{a}((\mathbf{bc})\mathbf{b}) +$$
$$\frac{1}{2}\left([\mathbf{a}^2, \mathbf{b}, \mathbf{c}] + [\mathbf{a}^2, \mathbf{b}, \mathbf{b}] + [\mathbf{a}^2, \mathbf{c}, \mathbf{b}] + [\mathbf{a}, \mathbf{b}^2, \mathbf{c}] + [\mathbf{a}, \mathbf{b}^2, \mathbf{b}] + [\mathbf{a}, \mathbf{c}^2, \mathbf{b}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}^2] + [\mathbf{a}, \mathbf{b}, \mathbf{b}^2] + [\mathbf{a}, \mathbf{c}, \mathbf{b}^2]\right) = 0$$

Using the conditions of **right alternativity** and

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{c}, \mathbf{b}] = 0$$

obtained from the expansion up to order three, only the first two terms "survive", ergo

$$((\mathbf{ab})\mathbf{c})\mathbf{b} = \mathbf{a}((\mathbf{bc})\mathbf{b})$$

That is the Bol identity is reproduced and we get no new condition in fourth order.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Free Algebra

Links:

- WIKIPEDIA - Free Algebra
- WIKIPEDIA - Free Object

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Graded Lie Algebra

A **Graded Lie Algebra** is a **Lie algebra** endowed with a gradation which is compatible with the Lie bracket. A graded Lie algebra is a **nonassociative graded algebra** under the bracket operation.

Papers:

- Graded Lie Algebras and q-commutative and r-associative Parameters - L. A. Wills-Toro, J. D. Vaelez, T. Craven pct. 1

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Griess Algebra

The **Griess Algebra** is the weight-2 subspace of the **Moonshine VOA** $V^{\natural}$. It is a non-associative but commutative algebra of dimension $196.884 = 196.883 + 1$ with a positive definite invariant bilinear form.

It has 48-dimensional associative subalgebras.

Since Griess's construction of the **Monster simple group** as the **automorphism group** of this algebra, many attempts have been made in order to better understand its nature.

Conway constructed a slightly modified version of it, called the **Conway-Griess Algebra**.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Group Ring

See also:

- **Ring**

Documents:

- Hyperkomplexe Grössen und Darstellungstheorie in Arithmetischer Auffassung (1927) - E. Noether local dct. 1

Links:

- WIKIPEDIA - Group Ring

Videos:

- Lecture I, The Group Algebra (2011) - A. Prasad

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Hamilton Loop

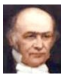
A **Hamiltonian Loop** is a (non-associative) **loop** for which all its subloops are **normal subloops**.

Such a loop is known to have a **modular lattice**.

Examples are *Cayley-Dickson loops* (see [1]).

Papers:

- Hamiltonian Loops (1949) - D. A. Norton local pct. 25
- [1] Automorphism Groups of Real Cayley-Dickson Loops (2011) - J. Kirshtein local pct. 0



Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Heisenberg Algebra

The **Heisenberg Algebra** (sometimes also called **Heisenberg-Weyl Algebra**) in **quantum mechanics** relates position- and momentum-operators and is defined by the following commutation-relations:

$$\begin{aligned} [\hat{x}^i, \hat{p}^j] &= i\hbar\delta^{ij} \\ [\hat{p}^i, \hat{p}^j] &= 0 \\ [\hat{x}^i, \hat{x}^j] &= 0 \end{aligned}$$

The first relation can be motivated as follows:

$$\begin{aligned} [\hat{x}^i, \hat{p}^j]\Phi(\mathbf{x}) &\equiv \left[\hat{x}^i, -i\hbar\frac{\partial}{\partial x^i}\right]\Phi(\mathbf{x}) \\ &= -i\hbar\left(\hat{x}^i\frac{\partial}{\partial x^i} - \delta^{ij} - \hat{x}^i\frac{\partial}{\partial x^i}\right)\Phi(\mathbf{x}) \end{aligned}$$

where in the last step we have used the chain rule of differentiation.

Generalizations

See

- **Noncommutative spacetime**

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Hentzel-Peresi Identity

The identities of degree ≤ 5 satisfied by **quadratic algebras** have been classified by Hentzel and Peresi.

Two of them satisfy the conditions:

1. they do not follow from the **flexibilty identity**, the **Jordan-** or the *Racine identities*,
2. together with these identities, they generate all the identities in degree 5 for the **sedenions**.

My own (preliminary) analysis however suggests, that one of them is trivial in the sense, that if one writes it out more explicitely, all terms cancel. Details of the analysis can be found *here*.

The other identity, which will be referred to as **Hentzel-Peresi Identity** is given as follows:

$$2[[\mathbf{A},\mathbf{B},\mathbf{A}],\mathbf{C},\mathbf{D}]+2[\mathbf{C},[\mathbf{A},\mathbf{B},\mathbf{A}],\mathbf{D}]-[\{[\mathbf{A},\mathbf{B}],\mathbf{A}\},\mathbf{C},\mathbf{D}]-[\{[\mathbf{A},\mathbf{B}],\mathbf{C}\},\mathbf{A},\mathbf{D}]+[[\mathbf{A},\mathbf{B}],\{\mathbf{A},\mathbf{C}\},\mathbf{D}]=0$$

Flexible algebra version

If the algebra is **flexible** (which is for example the case for **Cayley-Dickson algebras**) one has $[\mathbf{A},\mathbf{B},\mathbf{A}]=0$ and $\{[\mathbf{A},\mathbf{B}],\mathbf{A}\}=\mathbf{A}(\mathbf{AB})-(\mathbf{BA})\mathbf{A}$ and the identity simplifies to

$$[\mathbf{A}(\mathbf{AB})-(\mathbf{BA})\mathbf{A},\mathbf{C},\mathbf{D}]+[\{[\mathbf{A},\mathbf{B}],\mathbf{C}\},\mathbf{A},\mathbf{D}]-[[\mathbf{A},\mathbf{B}],\{\mathbf{A},\mathbf{C}\},\mathbf{D}]=0$$

or equivalently

$$[\mathbf{A}(\mathbf{AB}),\mathbf{C},\mathbf{D}]-[(\mathbf{BA})\mathbf{A},\mathbf{C},\mathbf{D}]+[\{[\mathbf{A},\mathbf{B}],\mathbf{C}\},\mathbf{A},\mathbf{D}]-[[\mathbf{A},\mathbf{B}],\{\mathbf{A},\mathbf{C}\},\mathbf{D}]=0$$

In terms of commutators and associators only, this reads:

$$[\mathbf{A}(\mathbf{AB}),\mathbf{C},\mathbf{D}]-[(\mathbf{BA})\mathbf{A},\mathbf{C},\mathbf{D}]+[[\mathbf{A},\mathbf{B}]\mathbf{C},\mathbf{A},\mathbf{D}]+[\mathbf{C}[\mathbf{A},\mathbf{B}],\mathbf{A},\mathbf{D}]-[[\mathbf{A},\mathbf{B}],\mathbf{AC},\mathbf{D}]-[[\mathbf{A},\mathbf{B}],\mathbf{CA},\mathbf{D}]=0$$

or with associators only

$$\begin{aligned}&[\mathbf{A}(\mathbf{AB}),\mathbf{C},\mathbf{D}]-[(\mathbf{BA})\mathbf{A},\mathbf{C},\mathbf{D}]+[(\mathbf{AB})\mathbf{C},\mathbf{A},\mathbf{D}]-[(\mathbf{BA})\mathbf{C},\mathbf{A},\mathbf{D}]+[\mathbf{C}(\mathbf{AB}),\mathbf{A},\mathbf{D}]-[\mathbf{C}(\mathbf{BA}),\mathbf{A},\mathbf{D}]\\&-[\mathbf{AB},\mathbf{AC},\mathbf{D}]+[\mathbf{BA},\mathbf{AC},\mathbf{D}]-[\mathbf{AB},\mathbf{CA},\mathbf{D}]+[\mathbf{BA},\mathbf{CA},\mathbf{D}]=0\end{aligned}$$

The polarized form is given by

$$\begin{aligned}&[\mathbf{A}(\mathbf{BC})-(\mathbf{CA})\mathbf{B},\mathbf{D},\mathbf{E}]+[\{[\mathbf{A},\mathbf{C}],\mathbf{D}\},\mathbf{B},\mathbf{E}]-[[\mathbf{A},\mathbf{C}],\{\mathbf{B},\mathbf{D}\},\mathbf{E}]+\\&[\mathbf{B}(\mathbf{AC})-(\mathbf{CB})\mathbf{A},\mathbf{D},\mathbf{E}]+[\{[\mathbf{B},\mathbf{C}],\mathbf{D}\},\mathbf{A},\mathbf{E}]-[[\mathbf{B},\mathbf{C}],\{\mathbf{A},\mathbf{D}\},\mathbf{E}]=0\end{aligned}$$

and again in terms of commutators and associators only by

$$\begin{aligned}&[\mathbf{A}(\mathbf{EB}),\mathbf{C},\mathbf{D}]+[\mathbf{E}(\mathbf{AB}),\mathbf{C},\mathbf{D}]-[(\mathbf{BA})\mathbf{E},\mathbf{C},\mathbf{D}]-[(\mathbf{BE})\mathbf{A},\mathbf{C},\mathbf{D}]+\\&[[\mathbf{A},\mathbf{B}]\mathbf{C},\mathbf{E},\mathbf{D}]+[[\mathbf{E},\mathbf{B}]\mathbf{C},\mathbf{A},\mathbf{D}]+[\mathbf{C}[\mathbf{A},\mathbf{B}],\mathbf{E},\mathbf{D}]+[\mathbf{C}[\mathbf{E},\mathbf{B}],\mathbf{A},\mathbf{D}]+\\&-[[\mathbf{A},\mathbf{B}],\mathbf{EC},\mathbf{D}]-[[\mathbf{E},\mathbf{B}],\mathbf{AC},\mathbf{D}]-[[\mathbf{A},\mathbf{B}],\mathbf{CE},\mathbf{D}]-[[\mathbf{E},\mathbf{B}],\mathbf{CA},\mathbf{D}]=0\end{aligned}$$

Explicit Form

1^{st} term:

$$\begin{aligned}&[\mathbf{A}(\mathbf{AB}),\mathbf{C},\mathbf{D}]-[(\mathbf{BA})\mathbf{A},\mathbf{C},\mathbf{D}]=\\&((\mathbf{A}(\mathbf{AB}))\mathbf{C})\mathbf{D}-(\mathbf{A}(\mathbf{AB}))(\mathbf{CD})-(((\mathbf{BA})\mathbf{A})\mathbf{C})\mathbf{D}+((\mathbf{BA})\mathbf{A})(\mathbf{CD})\end{aligned}$$

2^{nd} term:

$$\begin{aligned}&[(\mathbf{AB})\mathbf{C},\mathbf{A},\mathbf{D}]-[(\mathbf{BA})\mathbf{C},\mathbf{A},\mathbf{D}]+[\mathbf{C}(\mathbf{AB}),\mathbf{A},\mathbf{D}]-[\mathbf{C}(\mathbf{BA}),\mathbf{A},\mathbf{D}]=\\&(((\mathbf{AB})\mathbf{C})\mathbf{A})\mathbf{D}-((\mathbf{AB})\mathbf{C})(\mathbf{AD})-(((\mathbf{BA})\mathbf{C})\mathbf{A})\mathbf{D}+((\mathbf{BA})\mathbf{C})(\mathbf{AD})+\\&((\mathbf{C}(\mathbf{AB}))\mathbf{A})\mathbf{D}-(\mathbf{C}(\mathbf{AB}))(\mathbf{AD})-((\mathbf{C}(\mathbf{BA}))\mathbf{A})\mathbf{D}+(\mathbf{C}(\mathbf{BA}))(\mathbf{AD})\end{aligned}$$

3^{rd} term:

$$\begin{aligned}&-[\mathbf{AB},\mathbf{AC},\mathbf{D}]+[\mathbf{BA},\mathbf{AC},\mathbf{D}]-[\mathbf{AB},\mathbf{CA},\mathbf{D}]+[\mathbf{BA},\mathbf{CA},\mathbf{D}]=\\&-((\mathbf{AB})(\mathbf{AC}))\mathbf{D}+(\mathbf{AB})((\mathbf{AC})\mathbf{D})+((\mathbf{BA})(\mathbf{AC}))\mathbf{D}-(\mathbf{BA})((\mathbf{AC})\mathbf{D})+\\&-((\mathbf{AB})(\mathbf{CA}))\mathbf{D}+(\mathbf{AB})((\mathbf{CA})\mathbf{D})+((\mathbf{BA})(\mathbf{CA}))\mathbf{D}-(\mathbf{BA})((\mathbf{CA})\mathbf{D})\end{aligned}$$

One therefore has the following **association types** (6 out of the 14 possible of **degree 5**):

$$\begin{array}{lllll}(12,3): & ((\mathbf{BA})\mathbf{C})(\mathbf{AD}) & ((\mathbf{BA})\mathbf{A})(\mathbf{CD}) & ((\mathbf{AB})\mathbf{C})(\mathbf{AD}) & \\ (21,1): & ((\mathbf{A}(\mathbf{AB}))\mathbf{C})\mathbf{D} & ((\mathbf{C}(\mathbf{AB}))\mathbf{A})\mathbf{D} & ((\mathbf{C}(\mathbf{BA}))\mathbf{A})\mathbf{D} & \\ (2,1): & ((\mathbf{AB})(\mathbf{AC}))\mathbf{D} & ((\mathbf{AB})(\mathbf{CA}))\mathbf{D} & ((\mathbf{BA})(\mathbf{AC}))\mathbf{D} & ((\mathbf{BA})(\mathbf{CA}))\mathbf{D} \\ (3,1): & (((\mathbf{AB})\mathbf{C})\mathbf{A})\mathbf{D} & (((\mathbf{BA})\mathbf{A})\mathbf{C})\mathbf{D} & (((\mathbf{BA})\mathbf{C})\mathbf{A})\mathbf{D} & \\ (3,3): & (\mathbf{AB})((\mathbf{AC})\mathbf{D}) & (\mathbf{AB})((\mathbf{CA})\mathbf{D}) & (\mathbf{BA})((\mathbf{AC})\mathbf{D}) & (\mathbf{BA})((\mathbf{CA})\mathbf{D}) \\ (3,4): & (\mathbf{A}(\mathbf{AB}))(\mathbf{CD}) & (\mathbf{C}(\mathbf{AB}))(\mathbf{AD}) & (\mathbf{C}(\mathbf{BA}))(\mathbf{AD}) & \end{array}$$

Written out according to the lexicographical order of the elements occurring, the identity reads

$$\begin{aligned}&((\mathbf{A}(\mathbf{AB}))\mathbf{C})\mathbf{D}-(\mathbf{A}(\mathbf{AB}))(\mathbf{CD})+\\&-((\mathbf{AB})(\mathbf{AC}))\mathbf{D}+(\mathbf{AB})((\mathbf{AC})\mathbf{D})+\\&(((\mathbf{AB})\mathbf{C})\mathbf{A})\mathbf{D}-((\mathbf{AB})\mathbf{C})(\mathbf{AD})-((\mathbf{AB})(\mathbf{CA}))\mathbf{D}+(\mathbf{AB})((\mathbf{CA})\mathbf{D})+\\&-(((\mathbf{BA})\mathbf{A})\mathbf{C})\mathbf{D}+((\mathbf{BA})\mathbf{A})(\mathbf{CD})+((\mathbf{BA})(\mathbf{AC}))\mathbf{D}-(\mathbf{BA})((\mathbf{AC})\mathbf{D})+\\&-(((\mathbf{BA})\mathbf{C})\mathbf{A})\mathbf{D}+((\mathbf{BA})\mathbf{C})(\mathbf{AD})+((\mathbf{BA})(\mathbf{CA}))\mathbf{D}-(\mathbf{BA})((\mathbf{CA})\mathbf{D})+\\&((\mathbf{C}(\mathbf{AB}))\mathbf{A})\mathbf{D}-(\mathbf{C}(\mathbf{AB}))(\mathbf{AD})+\\&-((\mathbf{C}(\mathbf{BA}))\mathbf{A})\mathbf{D}+(\mathbf{C}(\mathbf{BA}))(\mathbf{AD})=0\end{aligned}$$

We observe that the position of $\mathbf{D}$ is fixed for all terms, whereas the other elements are reshuffled.

In terms of the association type labellings the identity reads

$$\begin{aligned}
&(21,1)-(3,4)+\\
&-(2,1)+(3,3)+\\
&(3,1)-(12,3)-(2,1)+(3,3)+\\
&-(3,1)+(12,3)+(2,1)-(3,3)+\\
&-(3,1)+(12,3)+(2,1)-(3,3)+\\
&(21,1)-(3,4)+\\
&-(21,1)+(3,4)=0
\end{aligned}$$

Defining

$$\begin{aligned}
\langle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\rangle &\equiv (((\mathbf{BA})\mathbf{C})\mathbf{A})\mathbf{D}-((\mathbf{BA})\mathbf{C})(\mathbf{AD})-((\mathbf{BA})(\mathbf{CA}))\mathbf{D}+(\mathbf{BA})((\mathbf{CA})\mathbf{D})-((\mathbf{C}(\mathbf{AB}))\mathbf{A})\mathbf{D}+(\mathbf{C}(\mathbf{AB}))(\mathbf{AD})\\
&= [(\mathbf{BA})\mathbf{C},\mathbf{A},\mathbf{D}]-[\mathbf{BA},\mathbf{CA},\mathbf{D}]-[\mathbf{C}(\mathbf{AB}),\mathbf{A},\mathbf{D}]\\
&= [(\mathbf{BA})\mathbf{C}-\mathbf{C}(\mathbf{AB}),\mathbf{A},\mathbf{D}]-[\mathbf{BA},\mathbf{CA},\mathbf{D}]\\
&\equiv \rangle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\langle-[\mathbf{BA},\mathbf{CA},\mathbf{D}]
\end{aligned}$$

this can be expressed according to

$$\begin{aligned}
\langle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\rangle-P_{\mathbf{A}\leftrightarrow\mathbf{B}}(\langle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\rangle)+P_{\mathbf{A}\leftrightarrow\mathbf{C}}(\langle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\rangle) &= -((\mathbf{AB})(\mathbf{AC}))\mathbf{D}+(\mathbf{AB})((\mathbf{AC})\mathbf{D})\\
&= -[\mathbf{AB},\mathbf{AC},\mathbf{D}]
\end{aligned}$$

or

$$\begin{aligned}
\rangle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\langle-P_{\mathbf{A}\leftrightarrow\mathbf{B}}(\rangle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\langle)+P_{\mathbf{A}\leftrightarrow\mathbf{C}}(\rangle\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\langle) &= -[\mathbf{AB},\mathbf{AC},\mathbf{D}]-[\mathbf{AB},\mathbf{CA},\mathbf{D}]+[\mathbf{BA},\mathbf{CA},\mathbf{D}]+[\mathbf{BA},\mathbf{AC},\mathbf{D}]\\
&= -[\mathbf{AB},\mathbf{AC},\mathbf{D}]-[\mathbf{AB},\mathbf{CA},\mathbf{D}]+[\mathbf{BA},\mathbf{CA},\mathbf{D}]+[\mathbf{BA},\mathbf{AC},\mathbf{D}]\\
&= -(1-P_{\mathbf{A}\leftrightarrow\mathbf{B}}+P_{\mathbf{A}\leftrightarrow\mathbf{C}}-P_{\mathbf{A}\leftrightarrow\mathbf{B}}P_{\mathbf{A}\leftrightarrow\mathbf{C}})([\mathbf{AB},\mathbf{AC},\mathbf{D}])
\end{aligned}$$

with $P_{\mathbf{A}\leftrightarrow\mathbf{B}}$ and $P_{\mathbf{A}\leftrightarrow\mathbf{C}}$ defining permutations of the appropriate elements.

An interpretation of this identity remains elusive. Yet it should take into account the fact that it is neither a sedenion- nor (necessarily) a **CD-algebra**-specific identity.
A better understanding of this identity is quite desirable, as it might be a generalisation of the **Jacobi identity**, which plays such an eminent role in physics.

(I have checked the "flexible" variant of the identity for the sedenions with **JHyperComplex**, and indeed, it is satisfied).

Papers:

- Identities for Algebras Obtained from the Cayley-Dickson Process (2001) - M. Bremner, I. Hentzel local pct. 4 prl. 10

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Icosian

The **Icosians** are a set of 120 unit **quaternions** forming the icosian **group** which is **isomorphic** to the **binary icosahedral group**.
The 120 elements are given by the union of the 24 **Hurwitz integers** and the following 96 quaternions which form the **root system** of H_4:

$$\begin{aligned}
&\frac{1}{2}\left(\pm\varphi\mathbf{e}\pm\mathbf{e}_i\pm\sigma\mathbf{e}_{i+1}\pm 0\mathbf{e}_{i+2}\right),\\
&\frac{1}{2}\left(\pm\mathbf{e}\pm\varphi\mathbf{e}_i\pm 0\mathbf{e}_{i+1}\pm\sigma\mathbf{e}_{i+2}\right),\\
&\frac{1}{2}\left(\pm\sigma\mathbf{e}\pm 0\mathbf{e}_i\pm\varphi\mathbf{e}_{i+1}\pm\mathbf{e}_{i+2}\right),\\
&\frac{1}{2}\left(\pm 0\mathbf{e}\pm\sigma\mathbf{e}_i\pm\mathbf{e}_{i+1}\pm\varphi\mathbf{e}_{i+2}\right),\quad i=1,2,3
\end{aligned}$$

where $\varphi=\frac{1}{2}\left(1+\sqrt{5}\right)$ and $\sigma=\frac{1}{2}\left(1-\sqrt{5}\right)$. (Note, that $\sigma^2+\varphi^2=3$). φ is the famous **Golden Ratio**.
The 0 was inserted for didactical reasons only.

The convex hull of the 120 unit icosians in 4-dimensional space form a *regular polychoron*, known as the *600-cell*.

The icosians are also a **ring**, known as **Icosian Ring**. It is abstractly **isomorphic** to the **E8 lattice**, three copies of which can be used to construct the **Leech lattice** using the *Turyn construction*.

Papers:

- A Highly Symmetric Four-dimensional Quasicrystal (1987) - V. Elser, N. J. A. Sloane local pct. 37
- Hyperbolic Weyl Groups and the four Normed Division Algebras (2008) - A. J. Feingold, A. Kleinschmidt, H. Nicolai local pct. 8
- Quaternionic Representation of Snub 24-Cell and its Dual Polytope Derived From E_8 Root System (2009) - M. Koca, M. Al-Ajmi, N. O. Koca local pct. 3
- Quaternionic and Octonionic Orbifolds (1989) - M. Koca local pct. 3
- E8 Lattice with Octonions and Icosians (1989) - M. Koca local pct. 0

Links:

- WIKIPEDIA - Icosian
- Icosians - K. G. S. Øyhus

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Jacobian

Given three elements $\mathbf{A}, \mathbf{B}, \mathbf{C}$ of an algebra, the **Jacobian (a.k.a Jacobiator)** $\mathbf{J}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ defines a cyclic ternary operation and is given by:

$$\mathbf{J}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \equiv (\mathbf{AB})\mathbf{C} + (\mathbf{BC})\mathbf{A} + (\mathbf{CA})\mathbf{B}$$

The identity $\mathbf{J}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = 0$ is called the **Jacobi Identity**. It holds if and only if the algebra is a **Lie algebra**.

We furthermore define a special case, called the **Commutator Jacobian**, with the product given by the **commutator product**:

$$\begin{aligned}\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) &\equiv [[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = -[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] - [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] - [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] \\ &= \frac{1}{2}\left([[\mathbf{A}, \mathbf{B}], \mathbf{C}] - [[\mathbf{B}, \mathbf{A}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] - [[\mathbf{C}, \mathbf{B}], \mathbf{A}]] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] - [[\mathbf{A}, \mathbf{C}], \mathbf{B}]\right)\end{aligned}$$

or

$$\mathbf{J}^c(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = 3[[\mathbf{A}_{[1}, \mathbf{A}_2], \mathbf{A}_{3]}]$$

It can equivalently be expressed in terms of the *cross product*:

$$\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) \equiv \vec{J}(\vec{A}, \vec{B}, \vec{C}) = 4\{(\vec{A} \times \vec{B}) \times \vec{C} + (\vec{B} \times \vec{C}) \times \vec{A} + (\vec{C} \times \vec{A}) \times \vec{B}\}$$

or in terms of **associators**:

$$\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) = [\mathbf{A}, \mathbf{B}, \mathbf{C}] - [\mathbf{A}, \mathbf{C}, \mathbf{B}] + [\mathbf{B}, \mathbf{C}, \mathbf{A}] - [\mathbf{B}, \mathbf{A}, \mathbf{C}] + [\mathbf{C}, \mathbf{A}, \mathbf{B}] - [\mathbf{C}, \mathbf{B}, \mathbf{A}]$$

Using the *Levi-Civita tensor* ε_{ijk} or **Bach brackets** this reads:

$$\begin{aligned}\mathbf{J}^c(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) &= \sum_{i,j,k=1\ldots3} \varepsilon_{ijk}[\mathbf{A}_i, \mathbf{A}_j, \mathbf{A}_k] \\ &= 6[\mathbf{A}_{[1}, \mathbf{A}_2, \mathbf{A}_{3]}]\end{aligned}$$

The Jacobi identity is still satisfied if one modifies each term by either 1 or 2 **involutions**, irrespective of which of the three elements it acts upon.

Examples

$$\begin{aligned}(\mathbf{A}^*\mathbf{B})\mathbf{C} + (\mathbf{B}^*\mathbf{C})\mathbf{A} + (\mathbf{C}\mathbf{A}^*)\mathbf{B} = 0 \\ (\mathbf{A}^*\mathbf{B}^*)\mathbf{C} + (\mathbf{B}\mathbf{C}^*)\mathbf{A}^* + (\mathbf{C}^*\mathbf{A}^*)\mathbf{B} = 0\end{aligned}$$

Properties

The commutator Jacobian is:

1. **linear** in all its components

$$\mathbf{J}^c(\sum_i \lambda_i \mathbf{A}_i, \sum_j \lambda_j \mathbf{B}_j, \sum_k \lambda_k \mathbf{C}_k) = \sum_{i,j,k} \lambda_i \lambda_j \lambda_k \mathbf{J}^c(\mathbf{A}_i, \mathbf{B}_j, \mathbf{C}_k)$$

2. **totally antisymmetric**:

$$\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) = -\mathbf{J}^c(\mathbf{A}, \mathbf{C}, \mathbf{B}) = -\mathbf{J}^c(\mathbf{B}, \mathbf{A}, \mathbf{C})$$

3. **cyclic**:

$$\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{J}^c(\mathbf{B}, \mathbf{C}, \mathbf{A}) = \mathbf{J}^c(\mathbf{C}, \mathbf{A}, \mathbf{B})$$

In coordinate form, the commutator Jacobian can be expressed as

$$\begin{aligned}\mathbf{J}^c(\mathbf{A}, \mathbf{B}, \mathbf{C}) &= J^c(\mathbf{A}, \mathbf{B}, \mathbf{C})^\sigma \mathbf{e}_\sigma \\ &= A_\mu B_\nu C_\rho J^c(\mathbf{e}_\mu, \mathbf{e}_\nu, \mathbf{e}_\rho)^\sigma \mathbf{e}_\sigma \\ &\equiv A_\mu B_\nu C_\rho J^c_{\mu\nu\rho}{}^\sigma \mathbf{e}_\sigma\end{aligned}$$

with $J^c_{\mu\nu\rho}{}^\sigma$ defining the **(Commutator) Jacobi Tensor**. Due to to the Jacobian being a tensorial object, we can use Latin and Greek indices equally well.

In terms of the **nonassociativity-tensor** we get

$$\begin{aligned}A_\mu B_\nu C_\rho J^{c\,\sigma}(\mathbf{e}_\mu, \mathbf{e}_\nu, \mathbf{e}_\rho)\mathbf{e}_\sigma &\equiv J^{c\,\sigma}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)\mathbf{e}_\sigma \\ &= 6[\mathbf{A}_{[1}, \mathbf{A}_2, \mathbf{A}_{3]}]^\sigma \mathbf{e}_\sigma \\ &= 12 A_\mu B_\nu C_\rho A^\sigma(\mathbf{e}_{[\mu}, \mathbf{e}_\nu, \mathbf{e}_{\rho]})\mathbf{e}_\sigma\end{aligned}$$

and hence

$$J^c_{\mu\nu\rho}{}^\sigma = 12 A_{[\mu\nu\rho]}{}^\sigma$$

TODO: The overall consistency of the factors of the various tensor relations is a mess (as it involves several definitions/conventions, differing in literature) and has yet to be established !

Furthermore it is related to the **torsion tensor** as follows:

$$\begin{aligned}J^c_{\mu\nu\rho}{}^\tau(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)\mathbf{e}_\tau &= 3[[\mathbf{A}_{[1}, \mathbf{A}_2], \mathbf{A}_{3]}] \\ &= 6 A_\mu B_\nu T^\sigma(\mathbf{e}_{[\mu}, \mathbf{e}_\nu)[\mathbf{e}_{|\sigma|}, \mathbf{A}_{3]}] \\ &= 12 A_\mu B_\nu C_\rho T^\sigma(\mathbf{e}_{[\mu}, \mathbf{e}_\nu) T^\tau(\mathbf{e}_{|\sigma|}, \mathbf{e}_{\rho]})\mathbf{e}_\tau\end{aligned}$$

Therefore

$$J_{\mu\nu\rho}^{c}{}^{\tau} - 12T^{\sigma}_{[\mu\nu}T^{\tau}_{|\sigma|\rho]} = 0$$

which is actually the general form of the **first Bianchi identity** or the **Akivis identity**.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Jordan Algebra

A **Jordan Algebra** is a **commutative** algebra (however generally not associative), satisfying the **Jordan identity**.
A Jordan algebra can be constructed from a non-commutative algebra by introducing a symmetric product $\circ$ called the **Jordan Product**:

$$\mathbf{A} \circ \mathbf{B} \equiv \frac{1}{2}(\mathbf{AB} + \mathbf{BA}) = \frac{1}{2}\{\mathbf{A}, \mathbf{B}\} = \frac{1}{4}\left[(\mathbf{A} + \mathbf{B})^2 - (\mathbf{A} - \mathbf{B})^2\right]$$

Any Jordan algebra (of characteristic $\neq 2$) is **power-associative**.

A weaker form of a Jordan algebra is a **Noncommutative Jordan Algebra**.

Theorem (Zelmanov)
Every simple Jordan algebra (of any dimension) is isomorphic to one of the following:

a. an algebra of a bilinear form,
b. an algebra of Hermitian type or
c. an **Albert algebra**.

Historical

- In 1933 Jordan suggested a new formulation of quantum mechanics based on Jordan algebras.
- In 1934 Jordan, von Neumann and Wigner showed that the Jordan algebras are always in one-to-one correspondence with a matrix algebra over the complex numbers with one exception, the exceptional Jordan algebra $\mathfrak{H}_3(\mathbb{O})$ of 3×3 matrices over the **octonions** (*TODO octonions or split octonions ???* . Later Albert proved that this is the only exceptional Jordan algebra the so called **Albert algebra**. The other Jordan algebras are called special.
- In 1978, Günaydin, Piron, and Ruegg showed [1] that it is possible to formulate quantum mechanics based on the exceptional Jordan algebra. It is called octonionic quantum mechanics. (The formulation of quantum mechanics with the special Jordan algebras brings about nothing new as it is equivalent to the Dirac formulation in terms of commutators).
- In 1983 Zelmanov accomplished the classification of infinite dimensional Jordan algebras. It appears that all infinite dimensional simple Jordan algebras are extensions of special Jordan algebras and that there are no infinite dimensional exceptional Jordan algebras. This implies that no **Hilbert space** formulation of octonionic quantum mechanics is possible.

Papers:

- On the Algebraic Structure of Quantum Mechanics (1967) - J. Gunson local pct. 69
- [1] Moufang Plane and Octonionic Quantum Mechanics (1978) - M. Günaydin, C. Piron, H. Ruegg local pct. 65
- Jordan Algebras and their Applications (1978) - K. McCrimmon local pct. 55
- On a Characterization of the State Space of Quantum Mechanics (1980) - H. Araki local pct. 54
- Quasi-Jordan Algebras (2006) - R. Velásquez, R. Felipe local pct. 20
- On Anti-Commutative Algebras with an Invariant Form (1962) - A. Sagle pct. 7

Lectures:

- Mini Course on Jordan Algebras - K. McCrimmon

Links:

- Jordan Theory Preprint Archives

Google books:

- Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics - H. Upmeier bct. 132

Videos:

- The Closest Cousins of Quantum Theory from Three Simple Principles (2012) - C. Ududec

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Jordan Identity

> ... a necessary law of any meaningful generalization of quantum mechanics.

- P. Jordan, J. Von Neumann, E. P. Wigner -

The **Jordan Identity** comes in 4 different versions, given by:

$$\begin{aligned}(\mathbf{AB})\mathbf{B}^2 &= (\mathbf{AB}^2)\mathbf{B} && (1)\\ \mathbf{A}^2(\mathbf{AB}) &= \mathbf{A}(\mathbf{A}^2\mathbf{B}) && (2)\\ (\mathbf{AB})\mathbf{A}^2 &= \mathbf{A}(\mathbf{BA}^2) \Leftrightarrow [\mathbf{A},\mathbf{B},\mathbf{A}^2]=0 && (3)\\ \mathbf{A}^2(\mathbf{BA}) &= (\mathbf{A}^2\mathbf{B})\mathbf{A} \Leftrightarrow [\mathbf{A}^2,\mathbf{B},\mathbf{A}]=0 && (4)\end{aligned}$$

Versions 2 and 3 are mentioned by Pascual Jordan in the context of a measurement algebra of quantum mechanics (see [1]).

If the algebra is **flexible** all 4 versions are equivalent.

As a mnemonic: Concerning the last two identities, the squared element is in either one of the outermost positions. Therefore theses identities can be expressed in terms of associators. In contrast, in case of the first two identities, the squared element changes the position within the identity.

A remarkable fact about the different versions is that, using the numbering system for **association types** of degree 4, one has

$$\begin{aligned}3 &= 4\\ 3 &= 2\\ 3 &= 5\\ 3 &= 1\end{aligned}$$

I.e. the different versions establish a relationship between association type 3 (which may be regarded as the "odd" one among the 5 association types - e.g. see **Sagle identity**) and all other possible association types.
This implies

$$((\mathbf{AA})\mathbf{A})\mathbf{A} = \mathbf{A}((\mathbf{AA})\mathbf{A}) = (\mathbf{AA})(\mathbf{AA}) = (\mathbf{A}(\mathbf{AA}))\mathbf{A} = \mathbf{A}(\mathbf{A}(\mathbf{AA}))$$

which can be interpreted as the degree 4 analogue of **monoassociativity**, which reads

$$(\mathbf{AA})\mathbf{A} = \mathbf{A}(\mathbf{AA})$$

Both relations are special cases of **power associativity**.

Association Type Expansions

By means of association type expansions one can obtain the constraints, the **nonassociativity tensor** has to satisfy.

1^{st} Jordan identity

Expansion of the identity

$$(\mathbf{BA})(\mathbf{AA}) = (\mathbf{B}(\mathbf{AA}))\mathbf{A}$$

up to third order yields

$$(\mathbf{ba})\mathbf{a} + (\mathbf{ba})\mathbf{a} + \mathbf{b}(\mathbf{aa}) + \mathbf{a}(\mathbf{aa}) - \mathbf{b}(\mathbf{aa}) - (\mathbf{ba})\mathbf{a} - (\mathbf{ba})\mathbf{a} - (\mathbf{aa})\mathbf{a}$$

or

$$[\mathbf{a},\mathbf{a},\mathbf{a}] = 0$$

This is the condition of monoassociativity.
Expressed in terms of the nonassociativity tensor it reads

$$A^{\sigma}_{(\mu\nu\rho)} = 0$$

2^{th} Jordan identity

Expansion of the identity

$$\mathbf{A}((\mathbf{AA})\mathbf{B}) = (\mathbf{AA})(\mathbf{AB})$$

up to third order yields

$$\mathbf{a}(\mathbf{aa}) + \mathbf{a}(\mathbf{ab}) + \mathbf{a}(\mathbf{ab}) + (\mathbf{aa})\mathbf{b} - (\mathbf{aa})\mathbf{a} - (\mathbf{aa})\mathbf{b} - \mathbf{a}(\mathbf{ab}) - \mathbf{a}(\mathbf{ab})$$

or

$$[\mathbf{a},\mathbf{a},\mathbf{a}]$$

which is the same result obtained for the first Jordan identity.

3^{nd} Jordan identity

Expansion of the identity

$$(\mathbf{AB})(\mathbf{AA}) = \mathbf{A}(\mathbf{B}(\mathbf{AA}))$$

up to third order yields

$$(\mathbf{ab})\mathbf{a} + (\mathbf{ab})\mathbf{a} + \mathbf{a}(\mathbf{aa}) + \mathbf{b}(\mathbf{aa}) - \mathbf{a}(\mathbf{ba}) - \mathbf{a}(\mathbf{ba}) - \mathbf{a}(\mathbf{aa}) - \mathbf{b}(\mathbf{aa}) = 0$$

or

$$2[\mathbf{a},\mathbf{b},\mathbf{a}] = 0$$

This is the flexibilty condition.
Expressed in terms of the nonassociativity tensor it reads

$$A^{\sigma}_{(\mu|\nu|\rho)} = 0$$

4^{rd} Jordan identity

Expansion of the identity

$$((\mathbf{AA})\mathbf{B})\mathbf{A} = (\mathbf{AA})(\mathbf{BA})$$

up to third order yields

$$(\mathbf{aa})\mathbf{b} + (\mathbf{aa})\mathbf{a} + (\mathbf{ab})\mathbf{a} + (\mathbf{ab})\mathbf{a} - (\mathbf{aa})\mathbf{b} - (\mathbf{aa})\mathbf{a} - \mathbf{a}(\mathbf{ba}) - \mathbf{a}(\mathbf{ba}) = 0$$

or

$$2[\mathbf{a}, \mathbf{b}, \mathbf{a}] = 0$$

which is the same result obtained for the third Jordan identity.

As two of the four Jordan identities imply flexibility, **noncommutative Jordan algebras** may be regarded as the more natural Jordan algebras. For the conventional **Jordan algebras** commutativity has to be introduced as an additional (ad hoc ?) condition.

Thus, considering a "measurement algebra" as specified by Pascual Jordan, the least one has to require from it is monoassociativity. Therefore **(hexagonal) 3-web** theory might serve as an adequate framework for an extended description of **quantum mechanics**. (Interestingly monoassociative 3-webs are **closed G-structures**, having maximally order four for any dimension).

Monoassociativity is a special case of flexibility. Hence if one requires flexibility of an algebra, all four identities are satisfied and equivalent, at least up to third order of their expansion.

Papers:

- On an Algebraic Generalization of the Quantum Mechanical Formalism (1934) - P. Jordan, J. Neumann, E. Wigner local pct. 457
- Power-Associative Rings (1947) - A. A. Albert local pct. 240 prl. 10
- Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik (1933) - P. Jordan local pct. 29
- A Moufang Loop, the Exceptional Jordan Algebra, and a Cubic Form in 27 Variables (1990) - R. L. Griess, Jr. local pct. 23
- [1] Über das Verhältnis der Theorie der Elementarlänge zur Quantentheorie (1968) - P. Jordan local pct. 7 prl. 10

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Jordan Triple Product

The **Jordan Triple Product** $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}_J$ is defined as

$$\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}_J = (\mathbf{AB}^*)\mathbf{C} + (\mathbf{CB}^*)\mathbf{A}?(\mathbf{AC})\mathbf{B}^*$$

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Kleinfeld Function

The **Kleinfeld Function f** is defined by:

$$\mathbf{f}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = [\mathbf{AB}, \mathbf{C}, \mathbf{D}] - \mathbf{B}[\mathbf{A}, \mathbf{C}, \mathbf{D}] - [\mathbf{B}, \mathbf{C}, \mathbf{D}]\mathbf{A}$$

More explicitly one has:

$$\mathbf{f}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = ((\mathbf{AB})\mathbf{C})\mathbf{D} - ((\mathbf{BC})\mathbf{D})\mathbf{A} + (\mathbf{B}(\mathbf{CD}))\mathbf{A} - \mathbf{B}((\mathbf{AC})\mathbf{D}) + \mathbf{B}(\mathbf{A}(\mathbf{CD})) - (\mathbf{AB})(\mathbf{CD})$$

i.e. it contains all 5 possible **association types** of degree 4.

Properties

- If the algebra is **alternative**, the Kleinfeld function is skew-symmetric in all of its arguments. Therefore it is zero if any two of its arguments are equal.

Fourfold cross product

In [1] the Kleinfeld function was used in the context of a generic fourfold *cross-product*, defined by

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \mathbf{D} \equiv \frac{1}{4}\left(\mathbf{A}^*(\mathbf{B} \times \mathbf{C} \times \mathbf{D}) + \mathbf{B}^*(\mathbf{C} \times \mathbf{D} \times \mathbf{A}) + \mathbf{C}^*(\mathbf{D} \times \mathbf{A} \times \mathbf{B}) + \mathbf{D}^*(\mathbf{B} \times \mathbf{A} \times \mathbf{C})\right)$$

If one takes $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathrm{Im}(\mathbb{O})$, with $\mathbb{O}$ the **octonion algebra**, one gets

$$\mathrm{Im}(\mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \mathbf{D}) = \frac{1}{4}\,\mathbf{f}(\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{C})$$

Papers:

- The Structure of Alternative Division Rings (1950) - R. H. Bruck, E. Kleinfeld local pct. 127

- Calibrated Geometries (1982) - R. Harvey, H. B. Lawson, Jr. local pct. 864 - The paper which started off the subject of **calibrated geometry**. prl. 10
- Alternative Division Rings of Characteristic 2 (1951) - E. Kleinfeld local pct. 21
- Cayley Integers (2005) - H. Holin local pct. 0

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Kleinfeld Identities

The **Kleinfeld Identities** are given by

$$\begin{aligned} [\mathbf{A},\mathbf{B}][[\mathbf{A},\mathbf{B}]^2,\mathbf{C},\mathbf{D}] = [\mathbf{A},\mathbf{B}]\,\mathbf{K}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = 0 \\ [[\mathbf{A},\mathbf{B}]^2,\mathbf{C},\mathbf{D}][\mathbf{A},\mathbf{B}] = \mathbf{K}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})\,[\mathbf{A},\mathbf{B}] = 0 \\ [[\mathbf{A},\mathbf{B}]^4,\mathbf{C},\mathbf{D}] = 0 \end{aligned}$$

where $\mathbf{K}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \equiv ([\mathbf{A},\mathbf{B}]^2,\mathbf{C},\mathbf{D})$ defines the so called **Kleinfeld Element**.

The identities are satisfied in any **alternative algebra**.

Papers:

- The Kleinfeld Identities in Generalized Accessible Rings (1976) - G. V. Dorofeev local pct. 1

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Lie Algebra

> ClassicalLie, remarkabLie, astroundingLie, physicalLie, undoubtedLie !
> SuperLie, naturalLie ? Hm ..., not realLie. Too critically ?
> - **Markus' Wisdom** -

A **Lie Algebra** $\mathfrak{g}$ over a **field** $\mathbb{K}$ is a vector space together with a bilinear map, $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the **Lie bracket**. It is required to satisfy

$$[\mathbf{A},\mathbf{A}] = 0,\ \forall \mathbf{A}\in\mathfrak{g}$$

as well as the **Jacobi identity**. Due to the relevant operation being a bilinear operation, a Lie algebra is a **Binary Algebra**.

A Lie algebra is the **tangent algebra** of a **Lie group** at the identity. (See also **Lie's theorems**).

A *semi-simple Lie algebra* is the sum of simple **ideals**. Therefore most of its properties can be obtained by first considering the latter ones. I.e. simple Lie algebras are the "building blocks" of the semi-simple ones.

A Lie algebra is specified by its **generators** $\mathbf{T}_i$ $(i = 1,\ldots,n)$ and their commutation relations

$$[\mathbf{T}_i,\mathbf{T}_j] = C_{ij}^k\mathbf{T}_k$$

with C_{ij}^k the *structure constants* of the algebra.
The **Dimension** of a Lie algebra is defined by the number of its generators n, spanning a n-dimensional vector space. This dimension is the same as that of the corresponding Lie group.

The "Magic Triangle" for Lie algebras. The "Magic Square" is framed by the double line

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $3\mathbf{A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $2\mathbf{A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $3\mathbf{A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

In **Clifford geometric algebra** the space of bivectors is closed under the **commutator product**, so it forms a Lie algebra (called a bivector algebra). Every Lie algebra is isomorphic to a bivector algebra.

Papers:

- Very Basic Lie Theory (1983) - R. Howe local pct. 66
- Exceptional Lie Algebras and Related Algebraic and Geometric Structures (1977) - J. R. Faulkner, J. C. Ferrar local pct. 46

Theses:

- Particle Dynamics of Branes (2008) - A. R. Ploegh

Videos:

- Introduction to Lie Algebras (2011) - R. Heluani
- Lie Groups Lie Algebras - Lectures 1-5 (2011) - F. Cachazo
- Lie Algebras and their Representations

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Lie Triple System

A finite dimensional vector space $\mathcal{V}$ over a field $\mathbb{K}$ equipped with a trilinear operation $(\ ,\ ,\ )$ is called a **Lie Triple System**, if for any $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$,$\mathbf{D}$,$\mathbf{E} \in \mathcal{V}$ the following identities hold:

$$\begin{aligned}(\mathbf{A},\mathbf{A},\mathbf{C}) &= 0 \Leftrightarrow (\mathbf{A},\mathbf{B},\mathbf{C}) = -(\mathbf{B},\mathbf{A},\mathbf{C})\\ \sigma_{\{A,B,C\}}(\mathbf{A},\mathbf{B},\mathbf{C}) &= 0 = (\mathbf{A},\mathbf{B},\mathbf{C}) + (\mathbf{B},\mathbf{C},\mathbf{A}) + (\mathbf{C},\mathbf{A},\mathbf{B})\\ (\mathbf{D},\mathbf{E},(\mathbf{A},\mathbf{B},\mathbf{C})) &= ((\mathbf{D},\mathbf{E},\mathbf{A}),\mathbf{B},\mathbf{C}) + (\mathbf{A},(\mathbf{D},\mathbf{E},\mathbf{B}),\mathbf{C}) + (\mathbf{A},\mathbf{B},(\mathbf{D},\mathbf{E},\mathbf{C}))\end{aligned}$$

Defining the map

$$\mathcal{D}_{\mathbf{D},\mathbf{E}}(\mathbf{X}) \equiv (\mathbf{D},\mathbf{E},\mathbf{X})$$

one can express the third relation according to

$$\mathcal{D}_{\mathbf{D},\mathbf{E}}((\mathbf{A},\mathbf{B},\mathbf{C})) = (\mathcal{D}_{\mathbf{D},\mathbf{E}}(\mathbf{A}),\mathbf{B},\mathbf{C}) + (\mathbf{A},\mathcal{D}_{\mathbf{D},\mathbf{E}}(\mathbf{B}),\mathbf{C}) + (\mathbf{A},\mathbf{B},\mathcal{D}_{\mathbf{D},\mathbf{E}}(\mathbf{C}))$$

$\mathcal{D}$ acts as a *derivation* and is also called **Inner** or **Interior Derivation**. It can be understood as a "ternary Leibnitz Rule".

A Lie triple system can be realised as a subspace of any **Lie algebra** which is closed under the ternary composition $[[\mathbf{A},\mathbf{B}],\mathbf{C}]$.

A special class of Lie triple systems are **Bol algebras**, satisfying additional binary bracket identities.

Moreover they arise as **tangent spaces** of smooth local **Bruck loops**.

The **tangent space** of a **symmetric space** forms a Lie triple system, since a symmetric space can be given the structure of a local **Bruck loop** at any point.

Papers:

- Application of Symmetric Spaces and Lie triple Systems in Numerical Analysis (2001) - H. Munthe-Kaas, G. R. W. Quispel, A. Zanna local pct. 8
- Ideals in Non-Associative Universal Enveloping Algebras of Lie Triple Systems (2005) - J.Mostovoy, J.M. Perez-Izquierdo local pct. 2
- Snyder Space-Time: K-Loop and Lie Triple System (2010) - F. Girelli local pct. 2
- Right Ideals in Non-associative Universal Enveloping Algebras of Lie Triple Systems (2006) - J. M. Pérez-Izquierdo local pct. 1
- A Universal Enveloping Algebra for a Lie Triple System (2005) - I. Burdujan local pct. 1
- A Generalization of Lie Groups and Symmetric Spaces (1986) - N. Hitotsuyanagi local pct. 0

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Magma

Links:

- WIKIPEDIA - Magma

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Malcev Algebra

An **algebra** (with a distributive multiplication) is called a **Malcev Algebra** if the following conditions are satisfied:

- Its elements *anticommute*, i.e. the **anti-commutator** satisfies $\{\mathbf{A},\mathbf{B}\} = 0$.
- The **Malcev identity** holds.

Examples

The following algebras are Malcev algebras:

- Every **Lie algebra**.
- Any **alternative algebra** equipped with the **commutator product**.

- The **imaginary octonions** $\mathbb{O}^-$ equipped with the commutator product (which is in fact the only simple compact Malcev algebra over $\mathbb{R}$ that is not a Lie algebra).

For a given Malcev algebra there existsts (up to **isomorphisms**) a uniquely determined connected and **simply connected** analytical **Moufang loop**. The Malcev algebra is the **tangent algebra** of this loop.

Documents:

- Speciality of Malcev Algebras (2005) - M. R. Bremner, I. R. Hentzel, L. A. Peresi local

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Malcev Identity

The **Malcev Identity** can be defined as:

$$((\mathbf{AB})\mathbf{C})\mathbf{A} + ((\mathbf{BC})\mathbf{A})\mathbf{A} + ((\mathbf{CA})\mathbf{A})\mathbf{B} - (\mathbf{AB})(\mathbf{AC}) = 0$$

It has the peculiarity that the first three elements are cyclic.

The linearized form of the Malcev identity is also known as **Sagle identity**.

The Malcev identity (or its linearized version) is regarded as a natural generalisation of the **Jacobi identity**.

In any **alternative algebra** the **Lie bracket** satisfies anticommutativity and the Malcev identity.

The Malcev identity in its form above is equivalent to the anticommutativity, **right- and middle Moufang** identities.

Proof:
Due to one of the middle Moufang identities one has

$$(\mathbf{AB})(\mathbf{CA}) = \mathbf{A}((\mathbf{BC})\mathbf{A})$$

Applying the anticommutativity condition to the left- and right hand site each leads to

$$(\mathbf{AB})(\mathbf{AC}) = ((\mathbf{BC})\mathbf{A})\mathbf{A}$$

Inserting this relationship into the Malcev identity above, two terms cancel and one is left with

$$((\mathbf{AB})\mathbf{C})\mathbf{A} + ((\mathbf{CA})\mathbf{A})\mathbf{B} = 0$$

which applying the antisymmetry condition three times reads

$$-((\mathbf{BA})\mathbf{C})\mathbf{A} + \mathbf{B}(\mathbf{A}(\mathbf{CA})) = 0$$

Yet this is nothing but the right Moufang identity.

Variants

Knowing the fact, that the Malcev identity can be decomposed into **association type identities** allows for the construction of variants of it. Firstly, one can apply the antisymmetry condition in various ways, leading to a couple of identities akin to the one above. This way one can "switch back and forth" between different **association types**.
E.g. if one prefers association type 5, one can get an identity in this respect by applying the antisymmetry condition twice to each of the first 3 terms

$$\mathbf{A}(\mathbf{C}(\mathbf{AB})) + \mathbf{A}(\mathbf{A}(\mathbf{BC})) + \mathbf{B}(\mathbf{A}(\mathbf{CA})) - (\mathbf{AB})(\mathbf{AC}) = 0$$

Secondly, we can replace the right Moufang condition by the left one and/or the one Middle Moufang identity by the other one.

As linearization leads to the Sagle identity, this also shows the origin of the different variants of this identity.

Jacobian

The Malcev identity can be expressed in terms of the **Jacobian** (if characteristic $\neq 2, 3$) as follows

$$\mathbf{J}(\mathbf{A}, \mathbf{B}, \mathbf{AC}) = \mathbf{J}(\mathbf{A}, \mathbf{B}, \mathbf{C})\mathbf{A}$$

or equivalently

$$\mathbf{J}(\mathbf{CA}, \mathbf{B}, \mathbf{A}) = \mathbf{A}\mathbf{J}(\mathbf{C}, \mathbf{B}, \mathbf{A})$$

Proof:
We proof the first identity. Written out this one reads:

$$(\mathbf{AB})(\mathbf{AC}) + (\mathbf{B}(\mathbf{AC}))\mathbf{A} + ((\mathbf{AC})\mathbf{A})\mathbf{B} = ((\mathbf{AB})\mathbf{C})\mathbf{A} + ((\mathbf{BC})\mathbf{A})\mathbf{A} + ((\mathbf{CA})\mathbf{B})\mathbf{A}$$

To have equivalence with the Malcev identity the following condition has to be satisfied:

$$((\mathbf{CA})\mathbf{B})\mathbf{A} - (\mathbf{B}(\mathbf{AC}))\mathbf{A} - ((\mathbf{AC})\mathbf{A})\mathbf{B} = ((\mathbf{CA})\mathbf{A})\mathbf{B}$$

which is, taking into consideration the anticommutativity of the product of a **Malcev algebra**.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Module

The concept of a **Module** is a generalization of the notion of a vector space. The difference is that a module is defined over a **ring**, whereas a vector space is defined over a **field**.

A module, like a vector space, is an additive **abelian group**; a product is defined between elements of the ring and elements of the module, and this multiplication is associative and distributive.

Definition

Let $\mathcal{R}$ be a ring and M an **abelian group**, then the structure of a **Left $\mathcal{R}$-module** on M is given by a function $\mathcal{R} \times M \to M$ such that for all $x, y \in \mathcal{R}$ and $m_1, m_2 \in M$,

$$\begin{aligned} x * (m_1 + m_2) &= x * m_1 + x * m_2 \\ x * (y * m_1) &= (xy) * m_1 \\ 1 * m_1 &= m_1 \end{aligned}$$

A **Right $\mathcal{R}$-module** is defined in a similar way.

In commutative ring theory, one generally deals once and for all either with left or right $\mathcal{R}$-modules.
However in noncommutative ring theory the situation is different: one regularly encounters modules of both types simultaneously.
Moreover, if $\mathcal{R}$ and $\mathcal{S}$ are rings we have the notion of an *R-S bimodule*.

Types of modules

- Finitely generated
- Cyclic
- Free
- Projective
- Injective
- Flat
- Simple
- Semisimple
- Indecomposable
- Faithful
- Noetherian
- Artinian
- Graded
- Uniform

For details, see [1].

Links:

- [1] WIKIPEDIA - Module (Mathematics)

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Monoassociativity

A **loop** is called **monoassociative** (or **3-power-associative**, short **3PA**) if each of its elements generates an associative subloop.
This implies that the **associators** of its local algebras satisfy the condition

$$[\mathbf{A}, \mathbf{A}, \mathbf{A}] = 0$$

A **quasigroup** is said to be local if all its local loops are monoassociative.
It follows that any tangent vector $\boldsymbol{\zeta}$ of such a local loop satisfies

$$[\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{\zeta}] = \zeta^\mu \zeta^\nu \zeta^\rho [\boldsymbol{e}_\mu, \boldsymbol{e}_\nu, \boldsymbol{e}_\rho] = \zeta^\mu \zeta^\nu \zeta^\rho A^\sigma_{\mu\nu\rho} \mathbf{e}_\sigma = 0$$

As

$$\zeta^\mu \zeta^\nu \zeta^\rho A^\sigma_{\mu\nu\rho} \mathbf{e}_\sigma = 0 = \zeta^\mu \zeta^\nu \zeta^\rho A^\sigma_{(\mu\nu\rho)} \mathbf{e}_\sigma$$

the monoassociativity condition in tensorial form reads

$$A^\sigma_{(\mu\nu\rho)} = 0$$

Monoassociativity is a weaker condition than is **power-associativity**.

Monoassociativity is akin to **alternativity** in that in the former case the associator vanishes if all three arguments are identical, whereas in the latter case the associator vanishes if two arguments are identical.
Thus monoassociativity is also weaker than is alternativity.

Example

Any **flexible algebra** is monoassociative, as it satisfies $[\mathbf{A}, \mathbf{B}, \mathbf{A}] = 0$.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Monster Group

The **Monster Group** $\mathbf{M}$ (or $\mathbf{F}_1$), which also goes under the names **Fischer-Griess Monster Group** or **The Friendly Giant Group** is the largest of the 26 **sporadic simple groups**, having order

$$\begin{aligned} & 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\ = \; & 808.017.424.794.512.875.886.459.904.961.710.757.005.754.368.000.000.000 \\ \approx \; & 8 \cdot 10^{53} \end{aligned}$$

Papers:

- The Friendly Giant (1982) - R. L. Griess, Jr. local pct. 266
- A Natural Representation of the Fischer-Griess Monster with the Modular Function J as Character (1984) - I. B. Frenkel, J. Lepowsky, A. Meurman local pct. 180
- A Simple Construction for the Fischer-Griess Monster Group (1985) - J. H. Conway local pct. 152
- On R. Griess' "Friendly Giant" (1984) - J. Tits local pct. 20
- Le Monstre (1983) - J. Tits local pct. 15
- Conjugacy Class Representatives in the Monster Group (2005) - R. W. Barraclough, R. A. Wilson local pct. 5
- Our Mathematical Universe: I. How the Monster Group Dictates All of Physics (2011) - F. Potter local pct. 2
- The Monster Sporadic Group and a Theory Underlying Superstring Models (1996) - G. Chapline local pct. 1
- A Conformally Invariant Approach to Estimation of Relations Between Physical Quantities (2007) - M. V. Gorbatenko, G. G. Kochemasov local pct. 1
- Conformal Geometrodynamics as the Basis for the Unified Description of Nature (2009) - M. V. Gorbatenko, G. G. Kochemasov local pct. 0
- Monster Sporadic Group Encoding of the Schwarzschild Metric (2004) - M. A. Thomas local pct. 0

Links:

- WIKIPEDIA - Monster Group
- Pushkinean Quantum Gravity (1983-1986) - K. Gennady

Videos:

- Monster Group
- Monster Group (a Little Extra Bit)

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Moufang Loop

A **Moufang loop** is an **algebra**, satisfying the following identities, called **Moufang Identities**:

Left Moufang Identity:

$$((\mathbf{AB})\mathbf{A})\mathbf{C} = \mathbf{A}(\mathbf{B}(\mathbf{AC}))$$

Right Moufang Identity:

$$((\mathbf{AB})\mathbf{C})\mathbf{B} = \mathbf{A}(\mathbf{B}(\mathbf{CB}))$$

Middle Moufang Identities:

$$\begin{aligned}(\mathbf{AB})(\mathbf{CA}) &= (\mathbf{A}(\mathbf{BC}))\mathbf{A} \\ \mathbf{A}((\mathbf{BC})\mathbf{A}) &= (\mathbf{AB})(\mathbf{CA})\end{aligned}$$

Therefore

$$\mathbf{A}((\mathbf{BC})\mathbf{A}) = (\mathbf{A}(\mathbf{BC}))\mathbf{A}$$

If one sets $\mathbf{C} = 1$ in the right Moufang identity one gets $[\mathbf{A}, \mathbf{B}, \mathbf{B}] = 0$, i.e. a Moufang loop is **right alternative**
Equivalently if one sets $\mathbf{B} = 1$ in the left Moufang identity one gets $[\mathbf{A}, \mathbf{A}, \mathbf{C}] = 0$, i.e. a Moufang loop is **left alternative**.
Setting $\mathbf{B} = 1$ in the first middle Moufang identity leads to $[\mathbf{A}, \mathbf{C}, \mathbf{A}] = 0$, i.e. a Moufang loop is **flexible**.

In fact for any **loop** the identities are equivalent, i.e. if one is satisfied the loop is a **flexible** and **alternative** algebra.

A loop is Moufang if and only if all of its **isotopes** are alternative.

As a Moufang loop is a left and a right **Bol loop** it also satisfies the left and right Bol loop identities.

Linearization

Linearizing the four Moufang identities one gets

$$\begin{aligned}\mathbf{M}_l(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv ((\mathbf{AB})\mathbf{C})\mathbf{D} - \mathbf{A}(\mathbf{B}(\mathbf{CD})) + ((\mathbf{CB})\mathbf{A})\mathbf{D} - \mathbf{C}(\mathbf{B}(\mathbf{AD})) = 0 \\ \mathbf{M}_r(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv \mathbf{A}(\mathbf{B}(\mathbf{CD})) - ((\mathbf{AB})\mathbf{C})\mathbf{D} + \mathbf{A}(\mathbf{D}(\mathbf{CB})) - ((\mathbf{AD})\mathbf{C})\mathbf{B} = 0 \\ \mathbf{M}_{m_1}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv \mathbf{A}(\mathbf{BC}))\mathbf{D} - (\mathbf{AB})(\mathbf{CD}) + \mathbf{D}(\mathbf{BC}))\mathbf{A} - (\mathbf{DB})(\mathbf{CA}) = 0 \\ \mathbf{M}_{m_2}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv \mathbf{A}((\mathbf{BC})\mathbf{D} - (\mathbf{AB})(\mathbf{CD}) + \mathbf{D}((\mathbf{BC})\mathbf{A} - (\mathbf{DB})(\mathbf{CA}) = 0\end{aligned}$$

and hence

$$\mathbf{M}_{m_1}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) - \mathbf{M}_{m_2}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \mathbf{A}(\mathbf{BC}))\mathbf{D} - \mathbf{A}((\mathbf{BC})\mathbf{D} + \mathbf{D}(\mathbf{BC}))\mathbf{A} - \mathbf{D}((\mathbf{BC})\mathbf{A} = 0$$

Introducing $\mathbf{Q}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \equiv ((\mathbf{AB})\mathbf{C})\mathbf{D} - \mathbf{A}(\mathbf{B}(\mathbf{CD}))$,

$$\begin{aligned}\mathbf{M}_l(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= \mathbf{Q}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) + \mathbf{Q}(\mathbf{C},\mathbf{B},\mathbf{A},\mathbf{D}) = 0 \\ \mathbf{M}_r(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= -\mathbf{Q}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) - \mathbf{Q}(\mathbf{A},\mathbf{D},\mathbf{C},\mathbf{B}) = 0\end{aligned}$$

i.e. $\mathbf{Q}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ exhibits the symmetries:

$$\mathbf{Q}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = -\mathbf{Q}(\mathbf{C},\mathbf{B},\mathbf{A},\mathbf{D}) = -\mathbf{Q}(\mathbf{A},\mathbf{D},\mathbf{C},\mathbf{B})$$

Examples

There are five nonassociative Moufang loops of **order** 16 (all of them are also **extra loops**). Among them is the Cayley loop (1845), which is the oldest known example of a nonassociative loop. The Cayley loop is usually described by starting with the **octonion** ring ($\mathbb{R}^8$), and restricting the multiplication to $\{\pm\mathbf{e}_i : 0 \le i \le 7\}$, where the $\mathbf{e}_i$ are the standard basis vectors. Restricting to $\mathbb{R}^8 \backslash \{0\}$ or to **S⁷** still yields a Moufang loop but not an extra loop any more.

People

Kuzmin, Kerdman, Sagle (and many others).

Papers:

- Zur Struktur von Alternativkörpern (1935) - R. Moufang local pct. 148
- The Varieties of Loops of Bol-Moufang Type (2007) - J. D. Phillips, P. Vojtechovsky local pct. 35
- An Introduction to Moufang Symmetry (1987) - E.N. Paal local pct. 8 - (in Russian)
- Extension of Local Loop Isomorphisms (1991) - P. T. Nagy local pct. 6
- Moufang Loops and Malcev Algebras (1992) - P. T. Nagy local pct. 3

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

N-ary Algebra

See also:

- **N-Quasigroup**
- **Ternary Algebra**

Papers:

- N-ary Algebras: A Review with Applications (2010) - J. A. de Azcárraga, J. M. Izquierdo local pct. 17

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Nonassociative Algebra

> ... algebras may be said to fly accurately through the dark like a bat, in constradistinction to geometry, which needs something more like the encompassing sight of the eagle to achieve its results, for it directly deals with substantive configurations and their modes of change. By its very nature, algebra tends to be more abstract, ...
> - Charles Musès -

In a **Nonassociative Algebra** $\mathcal{A}$ the relationship $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ for elements $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$ does not necessarily hold any more. This is equivalent to saying that the **associator** is not zero, i.e. $[\mathbf{A}, \mathbf{B}, \mathbf{C}] \neq 0$.

More generally, given a product of more than two elements, one has to distinguish between the different ways one can put parentheses, which is given by the **Catalan number** (see also **Association Type**).

Many concepts of associative algebras such as **subalgebra**, **ideal**, **homomorphism**, isomorphism, *simple algebra*, difference algebra, and direct sum do not involve associativity at all and can be taken over.

It is possible to construct a nonassociative algebra with almost any undesirable property. Most of these constructions are based on the use of a *multiplication table*.

It has been stated that no general theory of non-associative algebras exists yet.

People

M. A. Akivis, A. A. Albert, M. R. Bremner, A. S. Fedenko, A. N. Grishkov, I. R. Hentzel, M. Kikkawa, O. Kowalski, E. N. Kuz'min, A. I. Ledger, O. Loos, A. I. Mal'cev, P. Nagy, L. A. Peresi, L. V. Sabinin, A. A. Sagle, R. D. Schafer, I. P. Shestakov, K. Strambach (and many more ...).

Papers:

- Some Results in the Theory of Linear Non-associative Algebras (1944) - R. H. Bruck local pct. 54 TRD
- Non-Associative Tangles (1995) - D. Bar-Natan local pct. 54
- Nonassociative Algebras (2009) - M. R. Bremner, L. I. Murakami, I. P. Shestakov local pct. 12
- Structure and Representation of Nonassociative Algebras (1955) - R. D. Schafer local pct. 10 prl. 9
- Dimension Formulas for the Free Nonassociative Algebra (2005) - M. R. Bremner, I. R. Hentzel, L. A. Peresi local pct. 4 prl. 8
- Quelques Résultats D'algèbre Non Associative (1965) - M. Bertrand local pct. 0
- Algèbres Non-associatives et Applications à la Génétique (1963) - M. Bertrand local pct. 0

Links:

- Alberto Elduque
- Nonassociative Algebras Books

Google Books:

- A. Adrian Albert - Collected Mathematical Papers, Part 2: Nonassociative Algebras and Miscellany (1993) - N. Jacobson, J. M. Osborn, D. J. Saltman, D. Zelinsky, R. E. Block bct. 1

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Noncommutative Jordan Algebra

A **Noncommutative Jordan Algebra** per definitionem satisfies besides the **Jordan identity** the **flexibility law** (flexibility is a weaker condition than commutativity).

Properties

- Any noncommutative Jordan algebra of characteristic 0 is trace-admissible.
- All noncommutative Jordan algebras are **(strictly) power-associative**. The converse however is not true in general.
- Every flexible **quadratic algebra** is a noncommutative Jordan algebra.
- An algebra is a noncommutative Jordan algebra if and only if it is flexible and Jordan admissible (i.e. it can be made a **Jordan algebra** by symmetrization of the product).
- Noncommutative Jordan algebras include nil simple algebras, and fail to satisfy the *Wedderburn principal theorem* and the complete reducibility theorem. *TODO understand that*

The *simple* noncommutative Jordan algebras of characteristic 0 are

a. the simple (commutative) **Jordan algebras**,
b. the simple flexible *algebras of degree two*,
c. the simple *quasiassociative algebras*.

Examples

The classical **Cayley-Dickson algebras** are noncommutative Jordan algebras.

Historical

Noncommutative Jordan algebras were introduced by A. A. Albert.

Papers:

- Noncommutative Jordan Rings (1971) - K. McCrimmon local pct. 45
- Noncommutative Jordan Algebras of Characteristic 0 (1954) - R. D. Schafer local pct. 41
- (-1,-1)-Balanced Freudenthal Kantor Triple Systems and Noncommutative Jordan Algebras (2004) - A. A. Elduque, N. Kamiya, S. Okubo local pct. 12
- On a Class of Noncommutative Jordan Algebras (1966) - K. McCrimmon, R. D. Schafer local pct. 7
- A Note on Noncommutative Jordan Algebras (1957) - J. L. Page pct. 6
- Homotopes of Noncommutative Jordan Algebras (1971) - K. McCrimmon local pct. 4
- A Class of Non-commutative Jordan Algebra (1967) - C. E. Tsai local pct. 0

Abstracts:

- Power-associative Monocomposition Algebras - A. T. Gainov pct. 1

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Nondistributive Algebra

Papers:

- On the Notion of Lower Central Series for Loops - J. Mostovoy

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Normal Subloop

A subloop $\mathcal{N}$ of a **loop** $\mathcal{L}$ is called **Normal** (written $\mathcal{N} \triangleleft \mathcal{L}$) if, $\forall \mathbf{A}, \mathbf{B} \in \mathcal{L}$, it satisfies

$$\begin{aligned} \mathbf{A}\mathcal{N} &= \mathcal{N}\mathbf{A} &\Leftrightarrow\ & [\mathbf{A}, \mathcal{N}] = 0 &\Leftrightarrow\ & \mathbf{T}(\mathbf{A}) = \mathbf{L}_{\mathbf{A}}^{-1}\mathbf{R}_{\mathbf{A}} = \mathbf{e} \\ \mathbf{B}(\mathbf{A}\mathcal{N}) &= (\mathbf{B}\mathbf{A})\mathcal{N} &\Leftrightarrow\ & [\mathbf{B}, \mathbf{A}, \mathcal{N}] = 0 &\Leftrightarrow\ & \mathbf{L}(\mathbf{A}, \mathbf{B}) = \mathbf{L}_{\mathbf{AB}}^{-1}\mathbf{L}_{\mathbf{B}}\mathbf{L}_{\mathbf{A}} = \mathbf{e} \\ (\mathcal{N}\mathbf{A})\mathbf{B} &= \mathcal{N}(\mathbf{A}\mathbf{B}) &\Leftrightarrow\ & [\mathcal{N}, \mathbf{A}, \mathbf{B}] = 0 &\Leftrightarrow\ & \mathbf{R}(\mathbf{A}, \mathbf{B}) = \mathbf{R}_{\mathbf{AB}}^{-1}\mathbf{R}_{\mathbf{B}}\mathbf{R}_{\mathbf{A}} = \mathbf{e} \end{aligned}$$

$\mathbf{T}(\mathbf{A})$, $\mathbf{L}(\mathbf{A}, \mathbf{B})$ and $\mathbf{R}(\mathbf{A}, \mathbf{B})$ are called **Middle-**, **Left-**, and **Right-Inner Mappings** of a loop. $\mathbf{T}(\mathbf{A})$ is identical with *conjugation*.

$\mathcal{N}$ induces a **homomorphism** $L \to \mathcal{L}/\mathcal{N}$ exactly as in group theory. In particular, no nontrivial congruence can be defined on a loop having no proper normal subloop except for $\{\mathbf{e}\}$ and $\mathcal{L}$. Such a loop is said to be **Simple**.

See also: **Normal subgroup**.

Papers:

- The Complexity of Computing over Quasigroups - H. Caussinus, F. Lemieux local pct. 17 prl. 9

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Nucleus

The **Left-**, **Middle-** and **Right-Nucleus** of a **nonassociative algebra** $\mathcal{A}$ are defined as

$$\begin{aligned} N_l(\mathcal{A}) &\equiv \{\mathbf{X} \in \mathcal{A} : [\mathbf{X}, \mathcal{A}, \mathcal{A}] = 0\} \\ N_m(\mathcal{A}) &\equiv \{\mathbf{X} \in \mathcal{A} : [\mathcal{A}, \mathbf{X}, \mathcal{A}] = 0\} \\ N_r(\mathcal{A}) &\equiv \{\mathbf{X} \in \mathcal{A} : [\mathcal{A}, \mathcal{A}, \mathbf{X}] = 0\} \end{aligned}$$

respectively.

The **Nucleus** of $\mathcal{A}$ is defined as

$$\begin{aligned} N(\mathcal{A}) &\equiv N_l(\mathcal{A}) \cap N_m(\mathcal{A}) \cap N_r(\mathcal{A}) \\ &= \{\mathbf{X} \in \mathcal{A} : [\mathbf{X}, \mathbf{A}, \mathbf{B}] = [\mathbf{A}, \mathbf{X}, \mathbf{B}] = [\mathbf{A}, \mathbf{B}, \mathbf{X}] = 0, \forall \mathbf{A}, \mathbf{B} \in \mathcal{A}\} \end{aligned}$$

By means of the nucleus of a nonassociative algebra one can also define its **center**.

Properties

- If $\mathcal{A}$ is a **loop**, each of the nuclei $N_l(\mathcal{A})$, $N_m(\mathcal{A})$ and $N_r(\mathcal{A})$ is a subloop of $\mathcal{A}$.
- The nucleus of a loop is always an associative subloop and hence a **group**.
- If $\mathcal{A}$ is **alternative**, it is enough to require $N(\mathcal{A}) = N_l(\mathcal{A})$.
- If $\mathcal{A}$ is an **octonion** algebra over a **field** $\mathbb{F}$ then $N(\mathcal{A}) = \mathbb{F}$.

Papers:

- Nonassociative Algebras with Submultiplicative Bilinear Form (1994) - A. Cedilnik, B. Zalar local pct. 2

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Parastrophy

Every **quasigroup** belongs to a set of 6 quasigroups, called **Parastrophic** (by Sade), **Adjugate** (by Fisher and Yates) or **Conjugate** (by Stein).

Since parastrophisms and **isotopisms** are permutations, both form **groups**, the **Parastrophism Group** being **isomorphic** to the **symmetric group** S_3 or to one of its subgroups.

Papers:

- Isotopy and Parastrophy of Quasigroups (1962) - R. Artzy local pct. 13

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Parker Loop

The **Parker Loop** is a prominent example of a **code loop** and plays an important role in the theory of **sporadic finite simple groups**. It is obtained from the **Golay code**. The code can be described as the 12-dimensional subspace of $(\mathbb{Z}_2)^{24}$ given as the span of the rows of the following matrix:

$$G = \begin{pmatrix}
1 &&&&&&&&&&& & 1&0&1&0&0&0&1&1&1&0&1&1 \\
& 1 &&&&&&&&&& & 1&1&0&1&0&0&0&1&1&1&0&1 \\
&& 1 &&&&&&&&& & 0&1&1&0&1&0&0&0&1&1&1&1 \\
&&& 1 &&&&&&&& & 1&0&1&1&0&1&0&0&0&1&1&1 \\
&&&& 1 &&&&&&& & 1&1&0&1&1&0&1&0&0&0&1&1 \\
&&&&& 1 &&&&&& & 1&1&1&0&1&1&0&1&0&0&0&1 \\
&&&&&& 1 &&&&& & 0&1&1&1&0&1&1&0&1&0&0&1 \\
&&&&&&& 1 &&&& & 0&0&1&1&1&0&1&1&0&1&0&1 \\
&&&&&&&& 1 &&& & 0&0&0&1&1&1&0&1&1&0&1&1 \\
&&&&&&&&& 1 && & 1&0&0&0&1&1&1&0&1&1&0&1 \\
&&&&&&&&&& 1 & & 0&1&0&0&0&1&1&1&0&1&1&1 \\
&&&&&&&&&&& 1 & 1&1&1&1&1&1&1&1&1&1&1&0
\end{pmatrix}$$

Papers:

- A Series of Algebras Generalizing the Octonions and Hurwitz-Radon Identity (2011) - S. Morier-Genoud, V. Ovisenko local pct. 4 prl. 10

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Pierce Decomposition

Given an algebra $\mathcal{A}$, its **Pierce Decomposition** relative to an **idempotent** $\mathbf{E}$ is given by

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_{1/2} \oplus \mathcal{A}_1$$

with

$$\begin{aligned}
\mathcal{A}_k &= \{\mathbf{A} \in \mathcal{A} : \mathbf{AE} = \mathbf{EA} = k\mathbf{A}\}, \quad k = 0, 1 \\
\mathcal{A}_{1/2} &= \{\mathbf{A} \in \mathcal{A} : \mathbf{AE} + \mathbf{EA} = \mathbf{A}\}
\end{aligned}$$

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Poincaré-Birkhoff-Witt Theorem

The **Poincaré-Birkhoff-Witt (PBW) Theorem** implies that any **Lie algebra** $\mathcal{L}$ is **isomorphic** to a subalgebra of the commutator algebra of some associative algebra. This result is established by constructing an associative **universal enveloping algebra** $\mathcal{U}(\mathcal{L})$ for $\mathcal{L}$, together with an injective Lie algebra **homomorphism** $\mathcal{L} \to \mathcal{U}(\mathcal{L})$.
It follows that every identity satisfied by the **commutator** in every associative algebra is a consequence of **anticommutativity** and the **Jacobi identity**.

Theorem
Let $\mathcal{L}$ be a Lie algebra and let $\{\mathbf{e}_i\}$ with $i = 1, \dots, n$ be a basis of $\mathcal{L}$. Then the monomials $\mathbf{e}_1^{n_1} \otimes \dots \otimes e_n^{n_n}$ form a basis of the universal envelopping algebra $\mathcal{U}(\mathcal{L})$ of $\mathcal{L}$.

Generalizations

The theory of **Akivis-** and **Sabinin algebras** generalizes the theory of Lie algebras and their universal enveloping algebras.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Primitive Element

Let $\mathcal{A}$ be an **algebra** with unit $\mathbf{e}$ over a **field** $\mathbb{K}$ and assume that there exists an algebra **homomorphism**

$$\delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$$

An **Element** $\mathbf{P} \in \mathcal{A}$ is called **Primitive** with respect to δ (or simply δ-primitive) if

$$\delta(\mathbf{P}) = \mathbf{e} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{e}$$

In the associative case, the primitive elements coincide with the Lie polynomials, by a *theorem of Friedrichs*.

An **Almost Primitive Element** of a **free algebra** $\mathcal{F}$ is defined as an element that is not primitive in $\mathcal{F}$ but is primitive in any proper **subalgebra** of $\mathcal{F}$ containing it.

Papers:

- Dimension Formulas for the Free Nonassociative Algebra (2005) - M. R. Bremner, I. R. Hentzel, L. A. Peresi local pct. 8 prl. 8

Google books:

- Combinatorial Methods: Free Groups, Polynomials, and Free Algebras (2004) - A. A. Mikhalev, V. Shpilrain, J.-T. Yu bct. 43

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Quasigroup

> The studies of smooth quasigroups are motivated, in particular, by applications to physics (theory of anomalies, requiring a change of a group by loop).
> - L. V. Sabinin, L. V. Sbitneva, "Non-Associative Algebra and Its Applications" -

A **Quasigroup** Q can be defined in two equivalent ways:

I. It is a **magma** or **groupoid** (i.e. there existst a binary operation '$\cdot$') such that the equation $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ has a unique solution in Q given any two of the three elements of the equation. If Q possesses an element $\mathbf{E}$ satisfying $\mathbf{E} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{E} = \mathbf{A}$ for every $\mathbf{A} \in Q$, Q is called a **loop** with neutral element $\mathbf{E}$.

II. It is a nonempty set with three operations, multiplication '$\cdot$', left division '$\backslash$' and (right division) '$/$', such that

$$\mathbf{A}\backslash(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}, \quad \mathbf{A} \cdot (\mathbf{A}\backslash\mathbf{B}) = \mathbf{B}, \quad (\mathbf{A} \cdot \mathbf{B})/\mathbf{B} = \mathbf{A}, \quad (\mathbf{A}/\mathbf{B}) \cdot \mathbf{B} = \mathbf{A}$$

for any $\mathbf{A}, \mathbf{B} \in Q$. In the case that $\mathbf{A}/\mathbf{A} = \mathbf{B}\backslash\mathbf{B}$ for any $\mathbf{A}, \mathbf{B} \in Q$, Q is a **loop**.

A commonly used notation is:

$$\begin{aligned} \mathbf{A}\backslash\mathbf{B} &\equiv L_{\mathbf{A}}^{-1}(\mathbf{B}) \\ \mathbf{A}/\mathbf{B} &\equiv R_{\mathbf{B}}^{-1}(\mathbf{A}) \end{aligned}$$

Differentiable Quasigroups differ from **Lie groups** only by the fact that multiplication in quasigroups is not associative.

Since *multiplication tables* of finite quasigroups are precisely *Latin squares*, results obtained for Latin squares can be interpreted in the setting of finite quasigroups.

Historical

Interestingly, quasigroups in the form of *Latin squares* predate **groups** by several decades.

The notion of a quasigroup was introduced by Suschkewitsch in 1929 (see [1]).

Generalization

Classical quasigroups are based on a binary (2-ary) multiplication. A generalization to an n-ary multiplication are **n-quasigroups**.

Papers:

- Quasigroups II (1944) - A. A. Albert local pct. 180
- On the Foundations of Quasigroups (1956) - S. K. Stein local pct. 100
- [1] On a Generalization of the of the Associativity Law (1929) - A. Suschkewitsch local pct. 39
- Quasigroups (1988) - V. M. Galkin local pct. 7
- Loops and Quasigroups: Aspects of Current Work and Prospects for the Future (2000) - J. D.H. Smith local pct. 5
- Elements of Quasigroup Theory and some its Applications in Code Theory and Cryptology (2003) - V. Shcherbacov local pct. 2
- Valentin Danilovich Belousov (February 20, 1925, July 23, 1988) His Life and Work (2005) - G. B. Belyavskaya, W. A. Dudek, S. V. Shcherbacov local Belousov's Papers and Books local - In the opinion of many famous mathematicians Belousov was the leading specialist in the theory of quasigroups and loops during the sixties and seventies of the previous century.

Magazines:

- Quasigroups and Related Systems - With free access of older issues.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Quasigroup Isotopy

An **Isotopy** (or **Isotopism**) T in the context of **quasigroups** (including **loops**) is an ordered triple of bijections $T = (\alpha, \beta, \gamma)$ from one **quasigroup** Q to another quasigroup Q' satisfying

$$\alpha(\mathbf{A})\beta(\mathbf{B}) = \gamma(\mathbf{AB}) \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{Q}$$

It is a special case of a (quasigroup) homotopy for which the maps need not to be bijective. (Surjective and injective versions also exist, named epitopy and monotopy respectively).
If $Q = Q'$, the isotopy is called an **autotopy**.

An isotopy with $\gamma = 1$ is called a *principal isotopy*.
If α, β, γ are **orthogonal maps**, T is also referred to as an **Orthogonal Isotopy**.

If $T = (\alpha, \beta, \beta)$, then the triple is called a **Left Isotopism** and the quasigroups are called **Left Isotopes**.
If $T = (\alpha, \beta, \alpha)$, then the triple is called a **Right isotopism** and the quasigroups are called right isotopes.
If $T = (\alpha, \alpha, \beta)$, then the triple is called a middle isotopism and the groupoids are called middle isotopes.

Isotopisms are the appropriate **morphisms** for the study of quasigroups and **loops**.

The notion of isotopy was introduced by A. A. Albert.

Isotopies generalize the concept of an **isomorphism** (the latter is obtained if $\alpha = \beta = \gamma$). An isotopy is an **Equivalence Relation** for quasigroups and loops since it preserves the property of unique solvability. However it does not in general preserve identities. For example a loop, isotopic to a **flexible loop**, is not necessarily flexible. On the other hand the associativity identity and the **Moufang identity** are universal, i.e. an arbitrary loop, isotopic to a group or a Moufang loop, will also be a group or a Moufang loop respectively.
A quasigroup- or loop-identity is called a **Universal Identity** if it is invariant under a quasigroup or loop isotopy respectively. Any identity that is universal under quasigroup isotopy is also universal under loop isotopy.

An isotopy defines a bilinear map $\mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ given by

$$\mathbf{AB} \equiv \mathbf{A} \circ_{\alpha,\beta,\gamma} \mathbf{B} = \gamma^{-1}(\alpha(\mathbf{A})\beta(\mathbf{B}))$$

Properties

- If (α, β, γ) is an isotopy, then (β, γ, α) is also an isotopy.
- Isotopic **groups** are necessarily isomorphic. (Therefore the notion of isotopism is superfluous in group theory).
- If a loop is isotopic to a group, then it is isomorphic to that group and thus is itself a group.
- However, a quasigroup which is isotopic to a group need not be a group.
- In case of simple **Moufang loops** isotopisms and isomorphisms lead to the same equivalences.
- Every quasigroup is isotopic to a loop.
- In terms of *multiplication tables* (*Latin squares*), Q and Q' are isotopic if the multiplication table of Q' can be obtained from the multiplication table of Q by permuting the rows (by α), the columns (by β) and by renaming the elements (by γ): $\gamma(\mathbf{e}_i\mathbf{e}_j) = \gamma(C_{ij}^k\mathbf{e}_k) \to \alpha(\mathbf{e}_i)\beta(\mathbf{e}_j) = C_{ij}^k\gamma(\mathbf{e}_k)$. This is a more general situation than in the case of an isomorphim (i.e. $T = (\varphi, \varphi, \varphi)$). Then $\varphi(\mathbf{e}_i\mathbf{e}_j) = \varphi(C_{ij}^k\mathbf{e}_k) \to \varphi(\mathbf{e}_i)\varphi(\mathbf{e}_j) = C_{ij}^k\varphi(\mathbf{e}_k)$, thence the basis elements are merely renamed.

Generalizations

Given two sets equipped with n-ary operations (e.g. **n-ary products**)

$$\mu : X \times \ldots \times X \to X$$

and

$$\nu : Y \times \ldots \times Y \to Y$$

one can consider morphisms between them to be the morphisms between the corresponding $(n+1)$-ary relations, i.e. the $(n+1)$-tuples of maps $(\phi_1, \ldots, \phi_{n+1})$ from X to Y such that, for any $\mathbf{X}_1, \ldots, \mathbf{X}_n \in X$,

$$\nu(\phi_1(\mathbf{X}_1), \ldots, \phi_n(\mathbf{X}_n)) = \phi_{n+1}(\mu(\mathbf{X}_1, \ldots, \mathbf{X}_n))$$

In case of n-ary products, if we use the product notations "$\circ$" for ν and "$*$" for μ, we get

$$\phi_1(\mathbf{X}_1) \circ \phi_2(\mathbf{X}_2) = \phi_3(\mathbf{X}_1 * \mathbf{X}_2)$$

which reproduces the definition above in the case $n = 2$, when making the identifications $\mathbf{X}_1 = \mathbf{A}$, $\mathbf{B} = \mathbf{X}_2$ and $\phi_1 = \alpha$, $\phi_2 = \beta$, $\phi_3 = \gamma$.

Papers:

- Isotopy Invariants in Quasigroups (1970) - Falcooner local pct. 18
- Octonions, Simple Moufang Loops and Triality (2003) - G. P. Nagy, P. Vojtchovský local pct. 15

Abstracts:

- Orthogonal Isotopes of Monocomposition Algebras (1986) - A. T. Gainov local pct. 1

Lectures:

- Introduction to Algebra (2010) - M. Wodzicki local
- Lecture Notes by Bill Cherowitzo (chapter I.1.1 Quasigroups and Loops)

Links:

- WIKIPEDIA - Quasigroups
- WIKIPEDIA - Isotopy of Loops

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Quaternator

A **Quaternator** is a generalization of the **commutator** and the **associator** to three consecutive products. The name quaternator was coined by Akivis and Goldberg (in this context also the word **Alternator** was used). Related objects are **Associator Deviations** and **Coassociators**.

Quaternators are the **primitive elements** of a **free nonassociative algebra** in degree 4 which are not **Akivis elements**. Contrary to degrees 2 and 3 where there is only one operation, in degree 4 two fundamental operations show up. Yet, if the algebra is **monoassociative**, these two quaternators are interdependent [1].

Different definitions for the two quaternators can be found in literature (e.g. [2], [3], [4], [5], [6], [7]):

$$\begin{aligned}
-\mathbf{c}^l_2(\mathbf{C},\mathbf{A},\mathbf{D},\mathbf{B}) = \mathbf{p}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \mathbf{p}_{2,1}(\mathbf{A},\mathbf{B};\mathbf{C};\mathbf{D}) &\equiv [\mathbf{AB},\mathbf{C},\mathbf{D}] - \mathbf{A}[\mathbf{B},\mathbf{C},\mathbf{D}] - \mathbf{B}[\mathbf{A},\mathbf{C},\mathbf{D}] \\
&= ((\mathbf{AB})\mathbf{C})\mathbf{D} - (\mathbf{AB})(\mathbf{CD}) - \mathbf{A}((\mathbf{BC})\mathbf{D}) + \mathbf{A}(\mathbf{B}(\mathbf{CD})) - \mathbf{B}((\mathbf{AC})\mathbf{D}) + \mathbf{B}(\mathbf{A}(\mathbf{CD})) \\
&= 1 - 3 - 2 + 5 - 2 + 5 \\
\mathbf{q}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \mathbf{p}_{1,2}(\mathbf{A};\mathbf{B},\mathbf{C};\mathbf{D}) &\equiv [\mathbf{A},\mathbf{BC},\mathbf{D}] - \mathbf{B}[\mathbf{A},\mathbf{C},\mathbf{D}] - \mathbf{C}[\mathbf{A},\mathbf{B},\mathbf{D}] \\
&= (\mathbf{A}(\mathbf{BC}))\mathbf{D} - \mathbf{A}((\mathbf{BC})\mathbf{D}) - \mathbf{B}((\mathbf{AC})\mathbf{D}) + \mathbf{B}(\mathbf{A}(\mathbf{CD})) - \mathbf{C}((\mathbf{AB})\mathbf{D}) + \mathbf{C}(\mathbf{A}(\mathbf{BD})) \\
&= 4 - 2 - 2 + 5 + 2 + 5
\end{aligned}$$

The numbers designate the **association types** appearing.

In [2] the "left alternator" $\mathbf{c}^l_1(\mathbf{B},\mathbf{A},\mathbf{C},\mathbf{D}) = [\mathbf{A},\mathbf{B},\mathbf{CD}] - [\mathbf{A},\mathbf{B},\mathbf{C}]\mathbf{D} - [\mathbf{A},\mathbf{B},\mathbf{D}]\mathbf{C}$ is used instead of $\mathbf{q}$.
In [5] yet another variant, $\mathbf{p}_{1,2}(\mathbf{A};\mathbf{B},\mathbf{C};\mathbf{D}) = [\mathbf{A},\mathbf{BC},\mathbf{D}] - [\mathbf{A},\mathbf{B},\mathbf{C}]\mathbf{B} - [\mathbf{A},\mathbf{B},\mathbf{D}]\mathbf{C}$ appears.
In [6], [7] two slightly modified versions of quaternators are used, which are however equivalent modulo the **Akivis element** $\mathfrak{A}_2$. In order to be able to facilitate comparing results, we'll stick to these definitions, which are

$$\begin{aligned}
\mathfrak{Q}_1(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [\mathbf{AB},\mathbf{C},\mathbf{D}] - \mathbf{A}[\mathbf{B},\mathbf{C},\mathbf{D}] - [\mathbf{A},\mathbf{C},\mathbf{D}]\mathbf{B} \\
\mathfrak{Q}_2(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &\equiv [\mathbf{A},\mathbf{BC},\mathbf{D}] - \mathbf{B}[\mathbf{A},\mathbf{C},\mathbf{D}] - [\mathbf{A},\mathbf{B},\mathbf{D}]\mathbf{C}
\end{aligned}$$

(By the same argument, there is also an equivalence modulo an Akivis element to the (venerable) **Kleinfeld function**).

An interpretation (and mnemonic) is that the quaternators measure the deviation of the associators $[\cdots,\mathbf{C},\mathbf{D}]$ and $[\mathbf{A},\cdots,\mathbf{D}]$ from being *derivations*.

In the (frequent) case of a monoassociative algebra, we'll primarily use $\mathcal{Q}_1$ and define, in analogy to the notation of commutators and associators,

$$\begin{aligned}
[\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}] &\equiv \mathfrak{Q}_1(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \\
&= ((\mathbf{AB})\mathbf{C})\mathbf{D} - (\mathbf{AB})(\mathbf{CD}) - \mathbf{A}((\mathbf{BC})\mathbf{D} + \mathbf{A}(\mathbf{B}(\mathbf{CD})) - ((\mathbf{AC})\mathbf{D})\mathbf{B} + (\mathbf{A}(\mathbf{CD}))\mathbf{B}
\end{aligned}$$

Notably, this quaternator contains all 5 association types of degree 4, contrary to $\mathfrak{Q}_2$.

See also:

- **Quaternator identities**

Papers:

- [5] Algebras, Hyperalgebras, Nonassociative Bialgebras and Loops (2007) - J. M. Pérez-Izquierdo local pct. 37
- [3] On Hopf Algebra Structures over free Operads (2005) - R. Holtkamp local pct. 37
- [2] Classification of Multidimensional Three-webs According to Closure Conditions (1989) - A. M. Shelekhov (Russian) local pct. 8 prl. 10
- Fourth-order Alternators of a Local Analytic Loop and Three-webs of Multidimensional Surfaces (1989) - M. A. Akivis, A. M. Shelekhov local pct. 6 (Russian)
- [1] Enveloping Algebras of Malcev Algebras (2009) - M. R. Bremner, I. R. Hentzel, L. A. Peresi, M. V. Tvalavadze, H. Usefi local pct. 5
- Nilpotency and Dimension Series for Loops (2005) - J. Mostovoy local pct. 5
- [6] Polynomial Identities for Tangent Algebras of Monoassociative Loops (2011) - M. R. Bremner, S. Madrigada local pct. 4 prl. 10
- [4] Dimension Filtration on Loops (2004) - J. Mostovoy, J. M. Pérez-Izquierdo local pct. 4
- A Class of Isoclinic Three-webs (2008) - L. M. Pidzhakova (Russian) local pct. 0

Theses:

- [7] Envolventes Universales de Álgebras de Sabinin (2010) - S. M. Merino local - Page 156 -

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Quaternion

> Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you multiply triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them." But on the 16th day of the same month - which happened to be a Monday, and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely, $i^2 = j^2 = k^2 = ijk = -1$ which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away.
> - William Rowan Hamilton [1] -

Real algebras over $\mathbb{R}^4$, given by the set $\{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a,b,c,d \in \mathbb{R}; \mathbf{ij} = \mathbf{k}\}$, can be classified according to additional conditions they satisfy, namely

- **Quaternions** (Hamilton, 1843): $-\mathbf{i}^2 = -\mathbf{j}^2 = -\mathbf{k}^2 = 1$.
- **Bicomplex numbers (split quaternions)** (Segre, 1892): $\mathbf{i}^2 = -\mathbf{j}^2 = -\mathbf{k}^2 = 1$.
- **Biparacomplex Numbers** or **Coquaternions** (Cockle, 1848): $\mathbf{i}^2 = \mathbf{j}^2 = -\mathbf{k}^2 = 1$.
- Numbers based on the "hyperproduct": $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$.
- **Dual quaternions**: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0$.

The quaternions of unit **norm** form a **group isomorphic** to *SU(2)*. The finite subgroups of $SU(2)$ can be described by means of discrete elements of the unit quaternions. These are **binary polyhedral groups**, falling into five classes: The **cyclic groups**, the *dicyclic groups* (among them the **quaternion group** of order 8), the **binary tetrahedral group** of order 24, the **binary octahedral group** of order 48 and the **binary icosahedral group** of order 120.

MAGMA Help

```
// Non-split right-handed quaternion algebra defined via its 16 structure constants

Q:= Algebra<Rationals(), 4 |
[1,0,0,0 , 0,1,0,0 , 0,0,1,0 , 0,0,0,1],
[0,1,0,0 , -1,0,0,0, 0,0,0,1 , 0,0,-1,0],
[0,0,1,0 , 0,0,0,-1 , -1,0,0,0 , 0,1,0,0],
[0,0,0,1 , 0,0,1,0 , 0,-1,0,0 , -1,0,0,0]>;

e := Q.1; i := Q.2; j := Q.3; k := Q.4;

e*e; e*i; e*j; e*k;

i*i; j*j; k*k;

i*j; j*i; j*k; k*j; k*i; i*k;

i*j*k;

IsCommutative(Q);
IsAssociative(Q);
IsLie(Q);
```

Papers:

- Applications of Quaternions to Computation with Rotations (1979) - E. Salamin local pct. 41
- [1] A Brief History of Quaternions and of the Theory of Holomorphic Functions of Quaternionic Variables (2011) - A. Buchmann local pct. 2
- QUATERNION TRANSCENDENT FUNCTIONS AND DIFFERENTIAL OPERATORS (2007) - J. M. Machado, M. F. Borges local pct. 0

Lectures:

- The Quaternions and the Spaces S3, SU(2), SO(3), and RP3 local

Presentations:

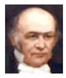
- From Algebras to Manifolds - R. Camarero, F. Etayo, C. G. Rovira, R. Santamaría local

Links:

- Quaternionic Physics

Videos:

- The Quaternion Handshake



Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Quaternion Multiplication Tables

In the following the *multiplication tables* for the **quaternions**, based on the classical **Cayley-Dickson doubling process**, are given for all 4 signatures. This leads to the "right-handed" quaternions.

CD(α,β)

This table encodes all the 4 possible tables.

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\alpha\mathbf{e}$	$\mathbf{k}$	$-\alpha\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$-\beta\mathbf{e}$	$\beta\mathbf{i}$
$\mathbf{k}$	$\alpha\mathbf{j}$	$-\beta\mathbf{i}$	$-\alpha\beta\mathbf{e}$

CD(+,+)

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\mathbf{e}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{e}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{e}$

CD(+,-)

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\mathbf{e}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{e}$	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$\mathbf{e}$

CD(-,+)

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{e}$	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{e}$	$\mathbf{i}$
$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{e}$

CD(-,-)

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{e}$	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{e}$	$-\mathbf{i}$
$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}$	$-\mathbf{e}$

The corresponding "left-handed" quaternion algebras can be obtained by flipping the signs of the off-diagonal elements of the multiplication tables. Hence:

CD(α,β)-L

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\alpha\mathbf{e}$	$-\mathbf{k}$	$\alpha\mathbf{j}$
$\mathbf{j}$	$\mathbf{k}$	$-\beta\mathbf{e}$	$-\beta\mathbf{i}$
$\mathbf{k}$	$-\alpha\mathbf{j}$	$\beta\mathbf{i}$	$-\alpha\beta\mathbf{e}$

CD(+,+)-L

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\mathbf{e}$	$-\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{e}$	$-\mathbf{i}$
$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}$	$-\mathbf{e}$

CD(+,-)-L

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\mathbf{e}$	$-\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{k}$	$\mathbf{e}$	$\mathbf{i}$
$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{e}$

CD(-,+)-L

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{e}$	$-\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{e}$	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$\mathbf{e}$

CD(-,-)-L

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{e}$	$-\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{k}$	$\mathbf{e}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{e}$

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sabinin Algebra

> This unappealing object turned out to be quite natural ...
> - J. M. Pérez-Izquierdo -

Sabinin Algebras (Hyperalgebras) are a nonassociative generalization of **Lie algebras** and their **universal enveloping algebras** in the following sense: the **tangent space** at the identity of any local **analytic loop** (without associativity assumptions) has a natural structure of a Sabinin algebra, and the classical correspondence between **Lie groups** and Lie algebras (see **Lie's theorems**) generalizes to this case.

One important thing about Sabinin algebras is that they are closely related to **affine connections** on a manifold $\mathcal{M}$. Every affine connection on $\mathcal{M}$ defined in some neighbourhood of the identity $\mathbf{e}$ determines a local multiplication at $\mathbf{e}$.
Conversely, each (not necessarily associative) local multiplication at $\mathbf{e}$ defines an affine connection on some neighbourhood of $\mathbf{e}$. This gives a one-to-one correspondence between affine connections and local analytic loops (a.k.a. infinitesimal loops). In other words, given a connection, the assumption of associativity for the algebraic description of $\mathcal{M}$ related to it is not justified in general. (A fact, hardly ever considered in the physics literature, although people at times talk about the "most general connection" ! Such a connection consists of a **Christoffel-**, **contorsion-** and **non-metricity-**part, i.e. at least one of them must do justice to the possibility of nonassociativity).

Although local nonassociative multiplications on manifolds can rarely be extended to global multiplications and, thus, cannot be studied directly by algebraic means, any local multiplication gives rise to an algebraic structure on the tangent space at $\mathbf{e}$, consisting of an infinite number of multilinear operations which has the structure of a Sabinin algebra. On the other hand, given a Sabinin algebra that satisfies certain convergence conditions, one can uniquely reconstruct the corresponding local analytic loop. Therefore, Sabinin algebras may be considered as the principal algebraic tool in studying local multiplications and local affine connections.

The set of the **primitive elements** of any **bialgebra** has the structure of a Sabinin algebra.
Every Sabinin algebra arises as the subalgebra of primitive elements in some **nonassociative Hopf algebra**.

Particular examples of Sabinin algebras besides Lie algebras are **Malcev-**, **Bol-**, Lie-Yamaguti-algebras as well as **Lie triple systems**.

In 2004 Pérez-Izquierdo has obtained a generalization of the **Poincaré-Birkhoff-Witt theorem** to Sabinin algebras.

Definition

A Sabinin algebra, denoted $(V, \langle \, ; \, \rangle, \mathbf{\Phi}(\, ; \,))$, is a vector space V equipped with 2 infinite families of multilinear operations $\langle \, ; \, \rangle$ and $\mathbf{\Phi}(\, ; \,)$, satisfying 4 identities, given as follows:

Multilinear operations

1.

$$\langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m; \mathbf{A}, \mathbf{B} \rangle, \;\; m \geq 0$$

$d \equiv m + 2$ will be called the d^{th} **Degree** of the algebra.

2.

$$\mathbf{\Phi}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n), \;\; m \geq 1, \, n \geq 2$$

$\mathbf{\Phi}$ is also known as **Multioperator**.

Identities

1.

$$\langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m; \mathbf{A}, \mathbf{B} \rangle = -\langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m; \mathbf{B}, \mathbf{A} \rangle \quad \text{(Anti-symmetry)}$$

2.

$$\langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r, \mathbf{Y}, \mathbf{Z}, \mathbf{X}_{r+1}, \ldots, \mathbf{X}_m; \mathbf{A}, \mathbf{B} \rangle - \langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r, \mathbf{Z}, \mathbf{Y}, \mathbf{X}_{r+1}, \ldots, \mathbf{X}_m; \mathbf{A}, \mathbf{B} \rangle + \sum_{k=0}^{r} \sum_{\sigma} \langle \mathbf{X}_{\sigma(1)}, \ldots, \mathbf{X}_{\sigma(k)}, \langle \mathbf{X}_{\sigma(k+1)}, \ldots, \mathbf{X}_{\sigma(r)}; \mathbf{Y}, \mathbf{Z} \rangle, \mathbf{X}_{r+1}, \ldots, \mathbf{X}_m; \mathbf{A}, \mathbf{B} \rangle = 0$$

$\forall \, r$, where σ is a $(k, r-k)$-**shuffle**.

3.

$$\sigma_{\{\mathbf{A},\mathbf{B},\mathbf{C}\}} \left(\langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r, \mathbf{A}; \mathbf{B}, \mathbf{C} \rangle + \sum_{k=0}^{r} \sum_{\sigma} \langle \mathbf{X}_{\sigma(1)}, \ldots, \mathbf{X}_{\sigma(k)}; \langle \mathbf{X}_{\sigma(k+1)}, \ldots, \mathbf{X}_{\sigma(r)}; \mathbf{B}, \mathbf{C} \rangle, \mathbf{A} \rangle \right) = 0$$

with $\sigma_{\{\mathbf{A},\mathbf{B},\mathbf{C}\}}$ designating the *cyclic sum*.

4.

$$\mathbf{\Phi}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n) = \mathbf{\Phi}(\mathbf{X}_{\sigma(1)}, \mathbf{X}_{\sigma(2)}, \ldots, \mathbf{X}_{\sigma(m)}, \mathbf{Y}_{\tilde{\sigma}(1)}, \mathbf{Y}_{\tilde{\sigma}(2)}, \ldots, \mathbf{Y}_{\tilde{\sigma}(n)})$$

with σ and $\tilde{\sigma}$ elements of the **symmetric group** acting on m and n elements respectively.

The operations $\langle \, ; \, \rangle$ and $\mathbf{\Phi}(\, ; \,)$ are independent and sometimes the term "Sabinin algebra" is used for a vector space equipped with the operations $\langle \, ; \, \rangle$ only, satisfying the corresponding properties. In many relevant cases the multioperator vanishes anyway.

In a remarkable paper Shestakov and Umirbaev [1] showed that a Sabinin algebra is closed under the multilinear operations and the commutator.

A **Sabinin Algebra of Degree** d (or **Degree** d **Sabinin Algebra**) with $d \geq 2$ is defined by considering only the multilinear operations of degree up to d, together with the polynomial identities of degree up to d involving these operations.

Construction

Examples

Historical

Sabinin algebras were introduced around 1986 by L. V. Sabinin and P. O. Mikheev.

Relationships

A local tangent space of a **loop manifold**, described by a Sabinin algebra, naturally can be interpreted as a **p-space** (devised by the author of this **Wiki-Blog**).
As Sabinin algebras code spaces with most general connections, it is not clear if the concept of a tangent space can be generalized further in a reasonable way. If this is not so, the concept of a P-space and that of a tangent space, described by a Sabinin algebra, supposedly coincide (which was only realised long after the introduction of P-spaces). Hence, P-spaces could be regarded as superfluous. At least Sabinin algebras describe a large class of P-spaces and presumably the relevant ones. Nevertheless, we'll stick to the concept of a P-space, just for the sake of having an own definition at hand, allowing for further modifications if this is required due to a better understanding of tangent structures. (Note, that the author of this Wiki-Blog is still lacking a good understanding of Sabinin algebras, odular structures etc., which are pretty difficult to digest).
Furthermore, the appropriate definition of a P-space is more guided by what is really needed in regards to applications to physics than by most general mathematical considerations. (I.e. it is intended to be kept as minimalistic as possible to most easily harness it for physics).

A personal remark

Given their fundamental nature, namely being consequent generalisations of Lie algebras, it is remarkable how little interest Sabinin algebras have stirred so far, in particular in respect to potential applications to physics.

Papers:

- [1] Free Akivis Algebras, Primitive Elements, and Hyperalgebras (2002) - I. P. Shestakov, U. U. Umirbaev local pct. 51
- Algebras, Hyperalgebras, Nonassociative Bialgebras and Loops (2007) - J. M. Pérez-Izquierdo local pct. 41
- On Hopf Algebra Structures over free Operads (2005) - R. Holtkamp local pct. 35
- Nonassociative Algebras - M. R. Bremner, L. I. Murakami, I. P. Shestakov local pct. 13
- Smooth Quasigroups and Geometry (1988) - P. O. Mikheev and L. V. Sabinin local pct. 11 prl. 10
- Formal Multiplications, Bialgebras of Distributions and Nonassociative Lie Theory (2009) - J. Mostovoy, J. M. Pérez-Izquierdo local pct. 10 prl. 9
- Ideals in Non-Associative Universal Enveloping Algebras of Lie Triple Systems (2005) - J.Mostovoy, J.M. Perez-Izquierdo local pct. 5
- Polynomial Identities for Tangent Algebras of Monoassociative Loops (2011) - M. R. Bremner, S. Madrigada local pct. 4 prl. 10
- Nilpotent Sabinin Algebras (2013) - J. Mostovoy, J. M. Perez-Izquierdo, I. P. Shestakov local pct. 1
- Hopf Algebras in Non-associative Lie Theory (2013) - J. Mostovoy, J. M. Pérez-Izquierdo, I. P. Shestakov local pct. 1
- Acerca de una Caracterización Algebraica de la Torsión de una Conexión Afín Plana (2010) - M. del Pilar Benito Clavijo, C. Jiménez Gestal, S. Madariaga Merino, J. M. Pérez local pct. 0

Theses:

- Envolventes Universales de Álgebras de Sabinin (2010) - S. M. Merino local

Lectures:

- Four Lectures on Formal Nonassociative Lie theory (2011) - J. M. Pérez-Izquierdo local

Videos:

- Non-associative Lie Theory (2013) - I. Shestakov local

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sabinin Algebra - Construction

Sabinin algebras can be realized recursively according to

$$\begin{aligned} \langle \mathbf{A}, \mathbf{B} \rangle &\equiv -[\mathbf{B}, \mathbf{A}], \quad \text{(commutator)} \\ \langle \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m, \mathbf{A}, \mathbf{B} \rangle &\equiv -\mathbf{p}_{m,1}(\mathbf{X}_1\mathbf{X}_2 \ldots \mathbf{X}_m, \mathbf{A}, \mathbf{B}) + \mathbf{p}_{m,1}(\mathbf{X}_1\mathbf{X}_2 \ldots \mathbf{X}_m, \mathbf{B}, \mathbf{A}) \end{aligned}$$

where $m \geq 1$, and

$$\mathbf{\Phi}_{m,n}(\mathbf{X}_1, \ldots, \mathbf{X}_m, \mathbf{Y}_1, \ldots, \mathbf{Y}_n) = \frac{1}{m!\,n!} \sum_{\sigma \in S_m} \sum_{\tau \in S_n} \mathbf{p}_{m,n-1}(\mathbf{X}_{\sigma(1)}, \ldots, \mathbf{X}_{\sigma(m)}, \mathbf{Y}_{\tau(1)}, \ldots, \mathbf{Y}_{\tau(n-1)}, \mathbf{Y}_{\tau(n)})$$

where $m \geq 1$ and $n \geq 2$. S_n and S_m denote the **symmetric group** of respective order.

Note that above and in the following all multiple products where brackets have been omitted are understood to be right-normed, that is, their **association type** is of the form $((\mathbf{AB})\ldots)$, or, in other words, with all opening brackets to the left of the first argument.

The polynomials $\mathbf{p}_{m,n}$ are defined by

$$\begin{aligned} \mathbf{p}_{0,0}(\mathbf{Z}) &= \mathbf{p}_{1,0}(\mathbf{X}_1, \mathbf{Z}) = \mathbf{p}_{0,1}(\mathbf{Y}_1, \mathbf{Z}) = \mathbf{0} \\ \mathbf{p}_{1,1}(\mathbf{X}_1, \mathbf{Y}_1, \mathbf{Z}) &= [\mathbf{X}_1, \mathbf{Y}_1, \mathbf{Z}], \quad \text{(associator)} \\ \mathbf{p}_{m,n}(\mathbf{X}_1 \cdots \mathbf{X}_m, \mathbf{Y}_1 \cdots \mathbf{Y}_n, \mathbf{Z}) &= [\mathbf{X}_1 \cdots \mathbf{X}_m,\, \mathbf{Y}_1 \cdots \mathbf{Y}_n,\, \mathbf{Z}] - \sum_{\substack{\mathbf{U}_1 \neq \mathbf{1} \text{ or } \mathbf{V}_1 \neq \mathbf{1} \\ \mathbf{U}_2 \neq \mathbf{1} \text{ and } \mathbf{V}_2 \neq \mathbf{1}}} \mathbf{U}_1 \mathbf{V}_1 \mathbf{p}_{m-\deg(\mathbf{U}_1), n-\deg(\mathbf{V}_1)}(\mathbf{U}_2, \mathbf{V}_2, \mathbf{Z})) \end{aligned}$$

$\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{V}_1$ and $\mathbf{V}_2$ result from so called 2-decompositions (for details, see [1]), given by

$$\begin{aligned} (\mathbf{U}_1, \mathbf{U}_2) &= (\mathbf{X}_{i_1} \cdots \mathbf{X}_{i_k}, \mathbf{X}_{i_{k+1}} \cdots \mathbf{X}_{i_m}) \\ (\mathbf{V}_1, \mathbf{V}_2) &= (\mathbf{Y}_{j_1} \cdots \mathbf{X}_{j_{k'}}, \mathbf{Y}_{j_{k'+1}} \cdots \mathbf{Y}_{j_n}) \end{aligned}$$

with the indices restricted to take the following values:

$$i_1 < \cdots < i_k, \qquad i_{k+1} < \cdots < i_m, \qquad I_1 \cup I_2 = \{1, \ldots, m\}, \qquad I_1 \cap I_2 = \emptyset$$

where $I_1 = \{i_1, \ldots, i_k\}$ and $I_2 = \{i_{k+1}, \ldots, i_m\}$ with $k \in \{1, \ldots, m\}$.
If $k = 0$ then $\mathbf{U}_1 = \mathbf{1}$; if $k = m$ then $\mathbf{U}_2 = \mathbf{1}$.
Analogously,

$$j_1 < \cdots < j_{k'}, \qquad j_{k'+1} < \cdots < j_n, \qquad I'_1 \cup I'_2 = \{1, \ldots, n\}, \qquad I'_1 \cap I'_2 = \emptyset$$

where $I'_1 = \{j_1, \ldots, j_{k'}\}$ and $I'_2 = \{j_{k'+1}, \ldots, j_n\}$ with $k' \in \{1, \ldots, n\}$.
If $k' = 0$ then $\mathbf{V}_1 = \mathbf{1}$; if $k' = n$ then $\mathbf{V}_2 = \mathbf{1}$.

Examples

Polynomials

$$\begin{aligned}
\mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}) &= [\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}] - \mathbf{Y}_1\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_2,\mathbf{Z}) - \mathbf{Y}_2\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z})\\
&= [\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}] - \mathbf{Y}_1[\mathbf{X}_1,\mathbf{Y}_2,\mathbf{Z}] - \mathbf{Y}_2[\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Z}]\\
\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}) &= [\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}] - \mathbf{X}_1\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}) - \mathbf{X}_2\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Z})\\
&= [\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}] - \mathbf{X}_1[\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}] - \mathbf{X}_2[\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Z}]
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}_{1,3}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3,\mathbf{Z}) &= [\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3,\mathbf{Z}] - \mathbf{Y}_1\mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_2\mathbf{Y}_3,\mathbf{Z}) - \mathbf{Y}_2\mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_3,\mathbf{Z}) - \mathbf{Y}_3\mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z})\\
&\quad - \mathbf{Y}_1\mathbf{Y}_2\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_3,\mathbf{Z}) - \mathbf{Y}_1\mathbf{Y}_3\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_3,\mathbf{Z}) - \mathbf{Y}_2\mathbf{Y}_3\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_2\mathbf{Y}_3,\mathbf{Z})\\
\mathbf{p}_{2,2}(\mathbf{X}_1\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}) &= [\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}] - \mathbf{X}_1\mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}) - \mathbf{X}_2\mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_1\mathbf{Y}_2,\mathbf{Z}) - \mathbf{Y}_1\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_2,\mathbf{Z}) - \mathbf{Y}_2\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z})\\
&\quad - \mathbf{X}_1\mathbf{Y}_1\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{Y}_2,\mathbf{Z}) - \mathbf{X}_1\mathbf{Y}_2\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}) - \mathbf{X}_2\mathbf{Y}_1\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_2,\mathbf{Z}) - \mathbf{X}_2\mathbf{Y}_2\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Z})\\
\mathbf{p}_{3,1}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3,\mathbf{Y}_1,\mathbf{Z}) &= [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3,\mathbf{Y}_1,\mathbf{Z}] - \mathbf{X}_1\mathbf{p}_{2,1}(\mathbf{X}_2\mathbf{X}_3,\mathbf{Y}_1,\mathbf{Z}) - \mathbf{X}_2\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_3,\mathbf{Y}_1,\mathbf{Z}) - \mathbf{X}_3\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z})\\
&\quad - \mathbf{X}_1\mathbf{X}_2\mathbf{p}_{1,1}(\mathbf{X}_3,\mathbf{Y}_1,\mathbf{Z}) - \mathbf{X}_1\mathbf{X}_3\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Z}) - \mathbf{X}_2\mathbf{X}_3\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Z})
\end{aligned}$$

Multilinear operations $\langle .,. \rangle$

Degree 3

$$\langle \mathbf{X}_1;\mathbf{A},\mathbf{B}\rangle = -\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{A},\mathbf{B}) + \mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{B},\mathbf{A})$$

Degree 4

$$\begin{aligned}
\langle \mathbf{X}_1,\mathbf{X}_2;\mathbf{A},\mathbf{B}\rangle &= -\mathbf{p}_{2,1}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{A},\mathbf{B}) + \mathbf{p}_{2,1}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{B},\mathbf{A})\\
&= -[\mathbf{X}_1\mathbf{X}_2,\mathbf{A},\mathbf{B}] + \mathbf{X}_1[\mathbf{X}_2,\mathbf{A},\mathbf{B}] + \mathbf{X}_2[\mathbf{X}_1,\mathbf{A},\mathbf{B}] + [\mathbf{X}_1\mathbf{X}_2,\mathbf{B},\mathbf{A}] - \mathbf{X}_1[\mathbf{X}_2,\mathbf{B},\mathbf{A}] - \mathbf{X}_2[\mathbf{X}_1,\mathbf{B},\mathbf{A}]
\end{aligned}$$

Degree 5

$$\begin{aligned}
\langle \mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3;\mathbf{A},\mathbf{B}\rangle &= -\mathbf{p}_{3,1}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{A},\mathbf{B}) + \mathbf{p}_{3,1}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{B},\mathbf{A})\\
&= -[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3,\mathbf{A},\mathbf{B}] + \mathbf{X}_1\mathbf{p}_{2,1}(\mathbf{X}_2\mathbf{X}_3,\mathbf{A},\mathbf{B}) + \mathbf{X}_2\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_3,\mathbf{A},\mathbf{B}) + \mathbf{X}_3\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_2,\mathbf{A},\mathbf{B})\\
&\quad + \mathbf{X}_1\mathbf{X}_2\mathbf{p}_{1,1}(\mathbf{X}_3,\mathbf{A},\mathbf{B}) + \mathbf{X}_1\mathbf{X}_3\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{A},\mathbf{B}) + \mathbf{X}_2\mathbf{X}_3\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{A},\mathbf{B})\\
&\quad [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3,\mathbf{B},\mathbf{A}] - \mathbf{X}_1\mathbf{p}_{2,1}(\mathbf{X}_2\mathbf{X}_3,\mathbf{B},\mathbf{A}) - \mathbf{X}_2\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_3,\mathbf{B},\mathbf{A}) - \mathbf{X}_3\mathbf{p}_{2,1}(\mathbf{X}_1\mathbf{X}_2,\mathbf{B},\mathbf{A})\\
&\quad - \mathbf{X}_1\mathbf{X}_2\mathbf{p}_{1,1}(\mathbf{X}_3,\mathbf{B},\mathbf{A}) - \mathbf{X}_1\mathbf{X}_3\mathbf{p}_{1,1}(\mathbf{X}_2,\mathbf{B},\mathbf{A}) - \mathbf{X}_2\mathbf{X}_3\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{B},\mathbf{A})
\end{aligned}$$

Multioperators

Degree 3

$$\begin{aligned}
\mathbf{\Phi}_{1,2}(\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Y}_2) &= \frac{1}{2}\left(\mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Y}_2) + \mathbf{p}_{1,1}(\mathbf{X}_1,\mathbf{Y}_2,\mathbf{Y}_1\right)\\
&= \frac{1}{2}\left([\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Y}_2] + [\mathbf{X}_1,\mathbf{Y}_2,\mathbf{Y}_1]\right)\\
&= [\mathbf{X}_1,\mathbf{Y}_{(1},\mathbf{Y}_{2)}]
\end{aligned}$$

Degree 4

$$\begin{aligned}
\mathbf{\Phi}_{1,3}(\mathbf{X}_1,\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3) &= \mathbf{p}_{1,2}(\mathbf{X}_1,\mathbf{Y}_{(1},\mathbf{Y}_2,\mathbf{Y}_{3)})\\
&= [\mathbf{X}_1,\mathbf{Y}_{(1}\mathbf{Y}_2,\mathbf{Y}_{3)}] - 2\mathbf{Y}_{(1}[\mathbf{X}_{|1|},\mathbf{Y}_2,\mathbf{Y}_{3)}]\\
\mathbf{\Phi}_{2,2}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Y}_2) &= \mathbf{p}_{2,1}(\mathbf{X}_{(1}\mathbf{X}_{2)},\mathbf{Y}_{(1},\mathbf{Y}_{2)})\\
&= [\mathbf{X}_{(1}\mathbf{X}_{2)},\mathbf{Y}_{(1},\mathbf{Y}_{2)}] - 2\mathbf{X}_{(1}[\mathbf{X}_{2)},\mathbf{Y}_{(1},\mathbf{Y}_{2)}]
\end{aligned}$$

where we have used **Bach bracket** notation.

See also:

- *Sabinin algebra - examples*
- **Akivis algebra**

Papers:

- [1] Free Akivis Algebras, Primitive Elements, and Hyperalgebras (2002) - I. P. Shestakov, U. U. Umirbaev local pct. 51 - A landmark paper -
- Algebras, Hyperalgebras, Nonassociative Bialgebras and Loops (2007) - J. M. Pérez-Izquierdo local pct. 38
- Polynomial Identities for Tangent Algebras of Monoassociative Loops (2011) - M. R. Bremner, S. Madrigada local pct. 4 prl. 10

Theses:

- Envolventes Universales de Álgebras de Sabinin (2010) - S. M. Merino local

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sagle Identity

The **Sagle Identity** is given by:

$$\mathbf{S}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \equiv ((\mathbf{AB})\mathbf{C})\mathbf{D} + ((\mathbf{BC})\mathbf{D})\mathbf{A} + ((\mathbf{CD})\mathbf{A})\mathbf{B} + ((\mathbf{DA})\mathbf{B})\mathbf{C} + (\mathbf{BD})(\mathbf{AC}) = \mathbf{0}$$

which is a linearized form of the **Malcev identity**.

As the first 4 terms are cyclic it can be expressed by means of the *cyclic sum*:

$$\mathbf{S}(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3,\mathbf{A}_4) \equiv \sigma_{\{1,2,3,4\}}((\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)\mathbf{A}_4 + (\mathbf{A}_2\mathbf{A}_4)(\mathbf{A}_1\mathbf{A}_3) = 0$$

The Sagle identity can be interpreted as a generalization of the **Jacobi identity** for quadruple products. We will therefore call $\mathbf{S}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ the **Saglean** in analogy to the **Jacobian**.

In terms of the commutator product this reads

$$\mathbf{S}^c(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \equiv [[[\mathbf{A},\mathbf{B}],\mathbf{C}],\mathbf{D}] + [[[\mathbf{B},\mathbf{C}],\mathbf{D}],\mathbf{A}] + [[[\mathbf{C},\mathbf{D}],\mathbf{A}],\mathbf{B}] + [[[\mathbf{D},\mathbf{A}],\mathbf{B}],\mathbf{C}] + [[\mathbf{B},\mathbf{D}],[\mathbf{A},\mathbf{C}]] = \mathbf{0}$$

where $\mathbf{S}^c(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ will be referred to as **Commutator Saglean**.
This can also be written by means of the two pure commutator **Akivis elements** $\mathcal{A}_1$ and $\mathcal{A}_3$ according to

$$\mathbf{S}^c(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \mathcal{A}_1(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) + \mathcal{A}_1(\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{A}) + \mathcal{A}_1(\mathbf{C},\mathbf{D},\mathbf{A},\mathbf{B})\mathcal{A}_1(\mathbf{D},\mathbf{A},\mathbf{B},\mathbf{C}) + \mathcal{A}_3(\mathbf{B},\mathbf{D},\mathbf{A},\mathbf{C}) = \mathbf{0}$$

thus neither **associators** nor **quaternators** are involved.

Changing the **association types** on the level of the commutator product for the first 4 terms, one gets the following identities

$$[\mathbf{A},[\mathbf{B},[\mathbf{C},\mathbf{D}]]] + [\mathbf{B},[\mathbf{C},[\mathbf{D},\mathbf{A}]]] + [\mathbf{C},[\mathbf{D},[\mathbf{A},\mathbf{B}]]] + [\mathbf{D},[\mathbf{A},[\mathbf{B},\mathbf{C}]]] + [[\mathbf{B},\mathbf{D}],[\mathbf{A},\mathbf{C}]] = \mathbf{0}$$

$$[\mathbf{A},[[\mathbf{B},\mathbf{C}],\mathbf{D}]] + [\mathbf{B},[[\mathbf{C},\mathbf{D}],\mathbf{A}]] + [\mathbf{C},[[\mathbf{D},\mathbf{A}],\mathbf{D}]] + [\mathbf{D},[[\mathbf{A},\mathbf{B}],\mathbf{C}]] + [[\mathbf{A},\mathbf{C}],[\mathbf{B},\mathbf{D}]] = \mathbf{0}$$

$$[[\mathbf{A},[\mathbf{B},\mathbf{C}]],\mathbf{D}] + [[\mathbf{B},[\mathbf{C},\mathbf{D}]],\mathbf{A}] + [[\mathbf{C},[\mathbf{D},\mathbf{A}]],\mathbf{B}] + [[\mathbf{D},[\mathbf{A},\mathbf{B}]],\mathbf{C}] + [[\mathbf{A},\mathbf{C}],[\mathbf{B},\mathbf{D}]] = \mathbf{0}$$

This is not an exhaustive list of possible expressions, as further combinations of terms with different association types are conceivable. (The expressions given above are just very symmetric ones). Further variants can be found in [1].

In characteristic $\neq 2,3$ the Saglean can be expressed by a linearized form of the Malcev identity in terms of the **Jacobian** as follows

$$\begin{aligned}\mathbf{S}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= \mathbf{J}(\mathbf{A}+\mathbf{D},\mathbf{B},(\mathbf{A}+\mathbf{D})\mathbf{C}) - \mathbf{J}(\mathbf{A}+\mathbf{D},\mathbf{B},\mathbf{C})(\mathbf{A}+\mathbf{D}) \\ &= \mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{AC}) + \mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{DC}) + \mathbf{J}(\mathbf{D},\mathbf{B},\mathbf{AC}) + \mathbf{J}(\mathbf{D},\mathbf{B},\mathbf{DC}) - \\ &\quad \mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{C})\mathbf{A} - \mathbf{J}(\mathbf{D},\mathbf{B},\mathbf{C})\mathbf{A} - \mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{C})\mathbf{D} - \mathbf{J}(\mathbf{D},\mathbf{B},\mathbf{C})\mathbf{D} \\ &= \mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{DC}) + \mathbf{J}(\mathbf{D},\mathbf{B},\mathbf{AC}) - \mathbf{J}(\mathbf{A},\mathbf{B},\mathbf{C})\mathbf{D} - \mathbf{J}(\mathbf{D},\mathbf{B},\mathbf{C})\mathbf{A}\end{aligned}$$

This expression can also be found in [2].

Analogously the commutator Saglean can be expressed in terms of the **commutator Jacobian** as follows

$$\mathbf{S}^c(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \mathbf{J}^c(\mathbf{A},\mathbf{B},\mathbf{DC}) + \mathbf{J}^c(\mathbf{D},\mathbf{B},\mathbf{AC}) - \mathbf{J}^c(\mathbf{A},\mathbf{B},\mathbf{C})\mathbf{D} - \mathbf{J}^c(\mathbf{D},\mathbf{B},\mathbf{CA}$$

If the algebra is **alternative**, the commutator Jacobian is proportional to the associator and one gets (suppressing a factor):

$$\mathbf{S}^c(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \equiv \mathbf{S}^c_{Alt}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = [\mathbf{A},\mathbf{B},\mathbf{DC}] + [\mathbf{D},\mathbf{B},\mathbf{AC}] - [\mathbf{A},\mathbf{B},\mathbf{C}]\mathbf{D} - [\mathbf{D},\mathbf{B},\mathbf{C}]\mathbf{A}$$

Thus the Sagle identity (in this variant) reads

$$[\mathbf{A},\mathbf{B},\mathbf{DC}] + [\mathbf{D},\mathbf{B},\mathbf{AC}] = [\mathbf{A},\mathbf{B},\mathbf{C}]\mathbf{D} + [\mathbf{D},\mathbf{B},\mathbf{C}]\mathbf{A}$$

Papers:

- [2] Simple Malcev Algebras over Fields of Characteristic Zero - A. A. Sagle local pct. 24
- [1] Invariant Nonassociative Algebra Structures on Irreducible Representations of Simple Lie Algebras - M. Bremner, I. Hentzel local pct. 12

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sedenion Loop

See also:

- **Quaternion group**
- **Octonion loop**
- *Cayley-Dickson loop*

Papers:

- C-Loops: An Introduction (2007) - J. D. Phillips, P. Vojtěchovský local pct. 18

Presentations:

- Subloops of Size 32 of Cayley-Dickson Loops (2012) - J. Kirshtein local

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sedenion Subalgebras

Although the underlying *projective spaces* are identical for the 16 different **sedenion algebras** obtained by a (classical) **Cayley-Dickson doubling process**, their **sign tables** and those of their subalgebras differ.

In the following we examine the subalgebra structure of the sedenion algebra $CD(1,1,1,1)$ and the two **split sedenion algebras** $CD(1,1,1,-1)$ and $CD(-1,1,1,1)$. These three algebras are not **isomorphic** to one another. (Most of the results were obtained with the help of the algebra program **JHyperComplex**).

Note, that the results are preliminary and incomplete yet (indicated by "?"s).

CD(1,1,1,1)

Basis: $\{\mathbf{e}, \mathbf{t}, \mathbf{x}, \mathbf{tx}, \mathbf{y}, \mathbf{ty}, \mathbf{xy}, (\mathbf{tx})\mathbf{y}, \mathbf{z}, \mathbf{tz}, \mathbf{xz}, (\mathbf{tx})\mathbf{z}, \mathbf{yz}, (\mathbf{ty})\mathbf{z}, (\mathbf{xy})\mathbf{z}, ((\mathbf{tx})\mathbf{y})\mathbf{z}\}$.

4-dimensional subalgebras

All algebras are of type $CD(+,+)$.

Nr.	Basis	Handedness
1	{e, t, y, ty}	Right
2	{e, x, yz, (xy)z}	Left
3	{e, tx, (ty)z, (xy)z}	R
4	{e, x, z, xz}	R
5	{e, y, xz, (xy)z}	R
6	{e, (tx)y, (tx)z, yz}	L
7	{e, tx, ty, xy}	L
8	{e, t, (xy)z, ((tx)y)z}	R
9	{e, ty, xz, ((tx)y)z}	R
10	{e, y, tz, (ty)z}	R
11	{e, (tx)y, tz, (xy)z}	R
12	{e, tx, y, (tx)y}	R
13	{e, x, ty, (tx)y}	R
14	{e, x, y, xy}	R
15	{e, xy, xz, yz}	L
16	{e, ty, (tx)z, (xy)z}	L
17	{e, t, xy, (tx)y}	L
18	{e, x, (ty)z, ((tx)y)z}	L
19	{e, x, tz, (tx)z}	R
20	{e, t, xz, (tx)z}	L
21	{e, (tx)y, xz, (ty)z}	L
22	{e, tx, tz, xz}	L
23	{e, t, z, tz}	R
24	{e, y, (tx)z, ((tx)y)z}	R
25	{e, xy, tz, ((tx)y)z}	L
26	{e, tx, yz, ((tx)y)z}	L
27	{e, t, x, tx}	R
28	{e, (tx)y, z, ((tx)y)z}	R
29	{e, xy, z, (xy)z}	R
30	{e, xy, (tx)z, (ty)z}	R
31	{e, y, z, yz}	R
32	{e, ty, tz, yz}	L
33	{e, tx, z, (tx)z}	R
34	{e, t, yz, (ty)z}	L
35	{e, ty, z, (ty)z}	R

Statistics

21 algebras are right-handed, 14 are left-handed.

8-dimensional subalgebras

Basis	8-dim. subalgebras	Blades	Signature	Type	Moufang	Sign table
{e, tx, ty, xy, tz, xz, yz, ((tx)y)z}	{7,9,15,22,25,26,32}	{1,0,6,0,1}	(1,7)	**Twisted**(+,+,+)	N	?
{e, tx, ty, xy, z, (tx)z, (ty)z, (xy)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(1,7)	CD(+,+,+)/**CD-like_24**(+,+,+)	Y	2
{e, t, xy, (tx)y, xz, (tx)z, yz, (ty)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(1,7)	Twisted(+,+,+)	N	?
{e, tx, y, (tx)y, tz, xz, (ty)z, (xy)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(1,7)	Unknown	N	?
{e, x, ty, (tx)y, tz, (tx)z, yz, (xy)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(1,7)	Unknown	N	?
{e, t, x, tx, yz, (ty)z, (xy)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(1,7)	CD-like 3(+,+,+)	N	?
{e, x, ty, (tx)y, z, xz, (ty)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(1,7)	CD(+,+,+)	Y	5
{e, t, y, ty, xz, (tx)z, (xy)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(1,7)	Unknown	N	?
{e, t, xy, (tx)y, z, tz, (xy)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(1,7)	CD(+,+,+)/CD-like_24(+,+,+)	Y	2
{e, tx, y, (tx)y, z, (tx)z, yz, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(1,7)	CD(+,+,+)	Y	5

{e, x, y, xy, tz, (tx)z, (ty)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(1,7)	Unknown	N	?
{e, t, x, tx, y, ty, xy, (tx)y}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(1,7)	CD(+,+,+)	Y	5
{e, x, y, xy, z, xz, yz, (xy)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(1,7)	CD(+,+,+)	Y	5
{e, t, y, ty, z, tz, yz, (ty)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(1,7)	CD(+,+,+)	Y	5
{e, t, x, tx, z, tz, xz, (tx)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(1,7)	CD(+,+,+)	Y	5

CD(1,1,1,-1)

Basis: $\{\mathbf{e}, \mathbf{x}, \mathbf{y}, \mathbf{xy}, \mathbf{z}, \mathbf{xz}, \mathbf{yz}, (\mathbf{xy})\mathbf{z}, \mathbf{t}, \mathbf{xt}, \mathbf{yt}, (\mathbf{xy})\mathbf{t}, \mathbf{zt}, (\mathbf{xz})\mathbf{t}, (\mathbf{yz})\mathbf{t}, ((\mathbf{xy})\mathbf{z})\mathbf{t}\}$.

4-dimensional subalgebras

Nr.	Basis	Type	Handedness
1	{e, xy, (xz)t, (yz)t}	CD(1,-1)	R
2	{e, x, yz, (xy)z}	CD(1,1)	L
3	{e, yz, xt, ((xy)z)t}	CD(1,-1)	L
4	{e, x, z, xz}	CD(1,1)	R
5	{e, y, xz, (xy)z}	CD(1,1)	R
6	{e, z, yt, (yz)t}	CD(1,-1)	R
7	{e, yz, t, (yz)t}	CD(1,-1)	R
8	{e, z, xt, (xz)t}	CD(1,-1)	R
9	{e, z, (xy)t, ((xy)z)t}	CD(1,-1)	R
10	{e, (xy)z, xt, (yz)t}	CD(1,-1)	R
11	{e, (xy)z, t, ((xy)z)t}	CD(1,-1)	R
12	{e, x, zt, (xz)t}	CD(1,-1)	L
13	{e, xy, zt, ((xy)z)t}	CD(1,-1)	L
14	{e, xz, (xy)t, (yz)t}	CD(1,-1)	L
15	{e, xy, t, (xy)t}	CD(1,-1)	R
16	{e, y, zt, (yz)t}	CD(1,-1)	L
17	{e, x, yt, (xy)t}	CD(1,-1)	L
18	{e, x, t, xt}	CD(1,-1)	R
19	{e, yz, yt, zt}	CD(1,-1)	L
20	{e, x, y, xy}	CD(1,1)	R
21	{e, xy, xz, yz}	CD(1,1)	L
22	{e, xz, yt, ((xy)z)t}	CD(1,-1)	R
23	{e, xz, t, (xz)t}	CD(1,-1)	R
24	{e, z, t, zt}	CD(1,-1)	R
25	{e, yz, (xy)t, (xz)t}	CD(1,-1)	R
26	{e, y, xt, (xy)t}	CD(1,-1)	R
27	{e, xy, xt, yt}	CD(1,-1)	L
28	{e, (xy)z, (xy)t, zt}	CD(1,-1)	L
29	{e, xz, xt, zt}	CD(1,-1)	L
30	{e, xy, z, (xy)z}	CD(1,1)	R
31	{e, y, z, yz}	CD(1,1)	R
32	{e, x, (yz)t, ((xy)z)t}	CD(1,-1)	R
33	{e, y, (xz)t, ((xy)z)t}	CD(1,-1)	L
34	{e, y, t, yt}	CD(1,-1)	R
35	{e, (xy)z, yt, (xz)t}	CD(1,-1)	L

Statistics

- 21 RH-algebras: $5 \times CD(1,1)$, $16 \times CD(1,-1)$.
- 14 LH-algebras: $2 \times CD(1,1)$, $12 \times CD(1,-1)$.

Algebras of type $CD(-1,\ldots)$ do not occur.

8-dimensional subalgebras

Basis	8-dim. subalgebras	Blades	Signature	Type	Moufang	Sign table
{e, xy, xz, yz, xt, yt, zt, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,0,6,0,1}	(5,3)	Twisted(+,+,-)	N	?
{e, xy, xz, yz, t, (xy)t, (xz)t, (yz)t}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	CD-like_24(+,+,-)	Y	?
{e, x, yz, (xy)z, yt, (xy)t, zt, (xz)t}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	Twisted(+,+,-)	N	?
{e, y, xz, (xy)z, xt, (xy)t, zt, (yz)t}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	Unknown	N	?
{e, xy, z, (xy)z, xt, yt, (xz)t, (yz)t}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	Unknown	N	?
{e, x, yz, (xy)z, t, xt, (yz)t, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD-like_24(+,+,-)	Y	?
{e, xy, z, (xy)z, t, (xy)t, zt, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD(+,+,-)	Y	?

{e, y, xz, (xy)z, t, yt, (xz)t, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD(+,+,-)	Y	?
{e, x, y, xy, zt, (xz)t, (yz)t, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD-like_3(+,+,-)	N	?
{e, y, z, yz, xt, (xy)t, (xz)t, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	Unknown	N	?
{e, x, z, xz, yt, (xy)t, (yz)t, ((xy)z)t}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	Unknown	N	?
{e, x, y, xy, z, xz, yz, (xy)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(1,7)	CD(+,+,+)	Y	5
{e, x, y, xy, t, xt, yt, (xy)t}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(5,3)	CD(+,+,-)	Y	?
{e, y, z, yz, t, yt, zt, (yz)t}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(5,3)	CD(+,+,-)	Y	?
{e, x, z, xz, t, xt, zt, (xz)t}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(5,3)	CD(+,+,-)	Y	?

CD(-1,1,1,1)

Basis: $\{\mathbf{e}, \mathbf{t}, \mathbf{x}, \mathbf{tx}, \mathbf{y}, \mathbf{ty}, \mathbf{xy}, (\mathbf{tx})\mathbf{y}, \mathbf{z}, \mathbf{tz}, \mathbf{xz}, (\mathbf{tx})\mathbf{z}, \mathbf{yz}, (\mathbf{ty})\mathbf{z}, (\mathbf{xy})\mathbf{z}, ((\mathbf{tx})\mathbf{y})\mathbf{z}\}$.

4-dimensional subalgebras

Nr.	Basis	Type	Handedness
1	{e, t, y, ty}	CD(-1,1)	R
2	{e, x, yz, (xy)z}	CD(1,1)	L
3	{e, tx, (ty)z, (xy)z}	CD(-1,-1)	L
4	{e, x, z, xz}	CD(1,1)	R
5	{e, y, xz, (xy)z}	CD(1,1)	R
6	{e, (tx)y, (tx)z, yz}	CD(-1,-1)	R
7	{e, tx, ty, xy}	CD(-1,-1)	R
8	{e, t, (xy)z, ((tx)y)z}	CD(-1,1)	R
9	{e, ty, xz, ((tx)y)z}	CD(-1,1)	R
10	{e, y, tz, (ty)z}	CD(1,-1)	R
11	{e, (tx)y, tz, (xy)z}	CD(-1,-1)	L
12	{e, tx, y, (tx)y}	CD(-1,1)	R
13	{e, x, ty, (tx)y}	CD(1,-1)	R
14	{e, x, y, xy}	CD(1,1)	R
15	{e, xy, xz, yz}	CD(1,1)	L
16	{e, ty, (tx)z, (xy)z}	CD(-1,-1)	R
17	{e, t, xy, (tx)y}	CD(-1,1)	L
18	{e, x, (ty)z, ((tx)y)z}	CD(1,-1)	L
19	{e, x, tz, (tx)z}	CD(1,-1)	R
20	{e, t, xz, (tx)z}	CD(-1,1)	L
21	{e, (tx)y, xz, (ty)z}	CD(-1,1)	L
22	{e, tx, tz, xz}	CD(-1,-1)	R
23	{e, t, z, tz}	CD(-1,1)	R
24	{e, y, (tx)z, ((tx)y)z}	CD(1,-1)	R
25	{e, xy, tz, ((tx)y)z}	CD(1,-1)	L
26	{e, tx, yz, ((tx)y)z}	CD(-1,1)	L
27	{e, t, x, tx}	CD(-1,1)	R
28	{e, (tx)y, z, ((tx)y)z}	CD(-1,1)	R
29	{e, xy, z, (xy)z}	CD(1,1)	R
30	{e, xy, (tx)z, (ty)z}	CD(1,-1)	R
31	{e, y, z, yz}	CD(1,1)	R
32	{e, ty, tz, yz}	CD(-1,-1)	R
33	{e, tx, z, (tx)z}	CD(-1,1)	R
34	{e, t, yz, (ty)z}	CD(-1,1)	L
35	{e, ty, z, (ty)z}	CD(-1,1)	R

Statistics

- 24 RH-algebras: $5 \times CD(1,1)$, $5 \times CD(1,-1)$ $9 \times CD(-1,1)$ $5 \times CD(-1,-1)$
- 11 LH-algebras: $2 \times CD(1,1)$, $2 \times CD(1,-1)$ $5 \times CD(-1,1)$ $2 \times CD(-1,-1)$

8-dimensional subalgebras

Basis	8-dim. subalgebras	Blades	Signature	Type	Moufang	Sign table
{e, tx, ty, xy, tz, xz, yz, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,0,6,0,1}	(5,3)	Unknown	N	?
{e, tx, ty, xy, z, (tx)z, (ty)z, (xy)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	CD(-,-,+)	Y	?
{e, x, ty, (tx)y, tz, (tx)z, yz, (xy)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	Unknown	N	?
{e, tx, y, (tx)y, tz, xz, (ty)z, (xy)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	Unknown	N	?
{e, t, xy, (tx)y, xz, (tx)z, yz, (ty)z}	{?,?,?,?,?,?,?}	{1,1,3,3,0}	(5,3)	Twisted(-,+,+)	N	?

{e, t, xy, (tx)y, z, tz, (xy)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD-like_24(-,+,+)	Y	?
{e, x, ty, (tx)y, z, xz, (ty)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD(+,-,+)	Y	?
{e, tx, y, (tx)y, z, (tx)z, yz, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	CD(-,+,+)	Y	?
{e, t, y, ty, xz, (tx)z, (xy)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	Unknown	N	?
{e, t, x, tx, yz, (ty)z, (xy)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	*CD-like_3*(-,+,+)	N	?
{e, x, y, xy, tz, (tx)z, (ty)z, ((tx)y)z}	{?,?,?,?,?,?,?}	{1,2,2,2,1}	(5,3)	Unknown	N	?
{e, x, y, xy, z, xz, yz, (xy)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(1,7)	CD(+,+,+)	Y	5
{e, t, y, ty, z, tz, yz, (ty)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(5,3)	CD(-,+,+)	Y	?
{e, t, x, tx, z, tz, xz, (tx)z}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(5,3)	CD(-,+,+)	Y	?
{e, t, x, tx, y, ty, xy, (tx)y}	{?,?,?,?,?,?,?}	{1,3,3,1,0}	(5,3)	CD(-,+,+)	Y	?

General results

- The sedenions contain 35 associative triads.
- 8 subalgebras are **Moufang loops**, the 7 others are not.
- A subalgebra is *norm-multiplicative* if and only if it is Moufang. This appears to be related to the **zero divisors** in the sedenions. Of the 14 octonary subalgebras that result from the last CD-doubling step, those that do not contain the imaginary unit related to that step ($\mathbf{z}$, $\mathbf{t}$, $\mathbf{z}$ respectively) are not norm-preserving. Any sedenion vector that is *pure* and therefore guaranteed to be not a zero divisor cannot be constructed exclusively from these algebras.
- **Twisted octonions** are necessarily a subalgebra of every type of sedenion [?].

Papers:

- Loops Embedded in Generalized Cayley Algebras of Dimension 2r, r = 2 (2000) - R. E. Cawagas local pct. 8
- On the Structure and Zero Divisors of the Cayley-Dickson Sedenion Algebra (2004) - R. E. Cawagas local pct. 7

Theses:

- Digraph Algebras over Discrete Pre-ordered Groups (2013) - K.-C. Chan local

Links:

- Sedenion Geometry - D. Chesley

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sedenion Zero Divisor

In the following we consider **zero divisors** of the non-split form of the (real) **sedenion** algebra obtained by a standard **Cayley-Dickson doubling process**. (For the split case, see **split sedenion zero divisor** and for non-split CD algebras in general, see *Cayley-Dickson algebra zero divisor*).

General properties

- The set of zero divisors in the sedenions (with entries of norm one) is **homeomorphic** to $Aut(\mathbb{O})$ = **G2**, the exceptional **Lie group** of rank two ([1]).
- An element $(\mathbf{A},\mathbf{B}) \in \mathbb{S}$ is a zero-divisor if and only if
  1. the **octonions** $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, i. e. $\langle \mathbf{A}|\mathbf{B}\rangle = 0$,
  2. both $\mathbf{A}$ and $\mathbf{B}$ are imaginary, i.e. $\Im(\mathbf{A}) = \mathbf{A}$, $\Im(\mathbf{B}) = \mathbf{B}$. In other words, $(\mathbf{A},\mathbf{B})$ is *doubly pure*.
  3. $\mathbf{A}$ and $\mathbf{B}$ are of equal length, i.e. $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{B})$.
- Geometrically the space of all unit zero divisors is homeomorphic to the **Stiefel manifold** $V_2(\mathbb{R}^7)$ of orthonormal pairs of vectors in $\mathbb{R}^7$. Here, $\mathbb{R}^7$ arises as the space of imaginary vectors in $\mathbb{A}_3$. *TODO This can be seen as follows:*
- Restricting the multiplication of $\mathbb{A}_4$ to $\mathbb{R}^9 \times \mathbb{R}^{16} \to \mathbb{R}^{16}$, where $\mathbb{R}^9 \subseteq \mathbb{R}^{16}$ is taken to be $\{(s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, 0, 0, 0, 0, 0, 0, 0); s_i \in \mathbb{R}, i = 0, \ldots, 8\}$, this restricted multiplication is **norm-preserving**.
- The same holds for products of the form $(\mathbf{A},\mathbf{0})(\mathbf{C},\mathbf{D})$, as due to the Cayley-Dickson doubling formula $(\mathbf{A},\mathbf{0})(\mathbf{C},\mathbf{D}) = (\mathbf{AC},\mathbf{DA})$. Therefore $(\mathbf{A},\mathbf{0})(\mathbf{C},\mathbf{D}) \equiv \mathbf{0} \Rightarrow \mathbf{AC} = \mathbf{0}, \mathbf{DA} = \mathbf{0}$ which implies that at least one of the elements $\mathbf{A}$, $\mathbf{C}$ or $\mathbf{D}$ must be non-zero, since $\mathbb{O}$ is a **division algebra**.

Classes of zero divisors

Simple zero divisors

There are 168 products of zero-divisors of the sedenions that are sums or differences of two units. (For details, see [2]). This coincides with the order of the **Klein group**.
Ignoring signs, there are 42 different pairings of units. These are referred to as **Assessors** in [3] (alluding to the "42 Assessors" of the Egyptian Book of the Dead, who sit in two rows of 21 along opposite walls of the Hall of Judgement).

Papers:

- [1] The Zero Divisors of the Cayley-Dickson Algebras over the Real Numbers (1997) - G. Moreno local pct. 33
- On the Structure and Zero Divisors of the Cayley-Dickson Sedenion Algebra (2004) - R. E. Cawagas local pct. 7
- [3] The 42 Assessors and the Box-Kites they fly: Diagonal Axis-Pair Systems of Zero-Divisors in the Sedenions' 16 Dimensions (2001) - R. P. C. de Marrais local pct. 2
- Quasilinear Actions on Products of Spheres (2010) - Ö. Ünlü, E. Yalçın local pct. 1
- The Vector Field Problem for Projective Stiefel Manifolds (2000) - J. Korbaš, P. Zvengrowski local pct. 0

Documents:

- Pure Spinors to Associative Triples to Zero-Divisors (2012) - F. D. Smith, Jr. - 2012 local

Links:

- [2] The 4*42=168 Products of Zero-divisors of the Sedenions that are Sums or Differences of Two Units - J. Arndt lrl. 9

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Sextenion

Papers:

- The Sextonions and E7 1/2 (2006) - J. M. Landsberg, L. Manivel local pct. 17

- Sextonions and the Magic Square (2004) - B. W. Westbury local pct. 8

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Spacetime Algebra

The **Spacetime Algebra** (**STA**) is the **Clifford algebra** *Cl(1,3)*$\simeq M_2(\mathbb{H})$, which is a 16-dimensional **algebra** spanned by the basis vectors

$$\begin{aligned}&\{\mathbf{e},\\ &\mathbf{e}_t,\ \mathbf{e}_x,\ \mathbf{e}_y,\ \mathbf{e}_z,\\ &\mathbf{e}_t\wedge\mathbf{e}_x,\ \mathbf{e}_t\wedge\mathbf{e}_y,\ \mathbf{e}_t\wedge\mathbf{e}_z,\ \mathbf{e}_x\wedge\mathbf{e}_y,\ \mathbf{e}_x\wedge\mathbf{e}_z,\ \mathbf{e}_y\wedge\mathbf{e}_z,\\ &\mathbf{e}_t\wedge\mathbf{e}_x\wedge\mathbf{e}_y,\ \mathbf{e}_t\wedge\mathbf{e}_x\wedge\mathbf{e}_z,\ \mathbf{e}_t\wedge\mathbf{e}_y\wedge\mathbf{e}_z,\ \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z,\\ &\mathbf{e}_t\wedge\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}\end{aligned}$$

Note that the algebras $Cl(1,3)$ and $Cl(3,1)$ (*Majorana algebra*) are not **isomorphic**. (See also: **Algebras and signature**). Nevertheless, both share the same even subalgebras: $Cl^+(1,3)\cong Cl^+(3,1)\cong M_2(\mathbb{C})$.

The Spacetime Algebra allows for an alternative (yet equivalent) **representation of the Dirac equation** dispensing with **matrices** altogether (see **Dirac-Hestenes Equation**).

Multiplication table

The following table was made using **JHyperComplex**, consequently applying the properties of the STA: Anticommutativity of the basis 1-vectors, $\mathbf{t}^2=\mathbf{e}$, $\mathbf{x}^2=\mathbf{y}^2=\mathbf{z}^2=-\mathbf{e}$ and associativity.

e	t	x	tx	y	ty	xy	(tx)y	z	tz	xz	(tx)z	yz	(ty)z	(xy)z	((tx)y)z
t	e	tx	x	ty	y	(tx)y	xy	tz	z	(tx)z	xz	(ty)z	yz	((tx)y)z	(xy)z
x	-tx	-e	t	xy	-(tx)y	-y	ty	xz	-(tx)z	-z	tz	(xy)z	-((tx)y)z	-yz	(ty)z
tx	-x	-t	e	(tx)y	-xy	-ty	y	(tx)z	-xz	-tz	z	((tx)y)z	-(xy)z	-(ty)z	yz
y	-ty	-xy	(tx)y	-e	t	x	-tx	yz	-(ty)z	-(xy)z	((tx)y)z	-z	tz	xz	-(tx)z
ty	-y	-(tx)y	xy	-t	e	tx	-x	(ty)z	-yz	-((tx)y)z	(xy)z	-tz	z	(tx)z	-xz
xy	(tx)y	y	ty	-x	-tx	-e	-t	(xy)z	((tx)y)z	yz	(ty)z	-xz	-(tx)z	-z	-tz
(tx)y	xy	ty	y	-tx	-x	-t	-e	((tx)y)z	(xy)z	(ty)z	yz	-(tx)z	-xz	-tz	-z
z	-tz	-xz	(tx)z	-yz	(ty)z	(xy)z	-((tx)y)z	-e	t	x	-tx	y	-ty	-xy	(tx)y
tz	-z	-(tx)z	xz	-(ty)z	yz	((tx)y)z	-(xy)z	-t	e	tx	-x	ty	-y	-(tx)y	xy
xz	(tx)z	z	tz	-(xy)z	-((tx)y)z	-yz	-(ty)z	-x	-tx	-e	-t	xy	(tx)y	y	ty
(tx)z	xz	tz	z	-((tx)y)z	-(xy)z	-(ty)z	-yz	-tx	-x	-t	-e	(tx)y	xy	ty	y
yz	tyz	(xy)z	((tx)y)z	z	tz	xz	(tx)z	-y	-ty	-xy	-(tx)y	-e	-t	-x	-tx
(ty)z	yz	((tx)y)z	(xy)z	tz	z	(tx)z	xz	-ty	-y	-(tx)y	-xy	-t	-e	-tx	-x
(xy)z	-((tx)y)z	-yz	(ty)z	xz	-(tx)z	-z	tz	-xy	(tx)y	y	-ty	-x	tx	e	-t
((tx)y)z	-(xy)z	-(ty)z	yz	(tx)z	-xz	-tz	z	-(tx)y	xy	ty	-y	-tx	x	t	-e

In order to facilitate comparison with the **multiplication tables of the sedenion algebras**, which were generated by means of the **Cayley-Dickson doubling process**, the same ordering of basis elements was chosen. This reveals that the underlying Abelian group structure is identical, corresponding with the *projective geometry* **PG(3,2)**. (This is demonstrated for the sedenion algebras under **sedenion subalgebras**). The differences are found on the "**cohomological** level", i.e. the values of the *structure constants* differ. In fact, the differences boil down to different signs of the structure constants. These differences are are marked in red.

Permuting rows and columns appropriately leads to the multiplication table found under [1], which is the same (up to relabellings) as one I constructed many years ago independently (see [2]). This table looks as follows:

e	t	x	y	z	tx	ty	tz	xy	xz	yz	(tx)y	(tx)z	(ty)z	(xy)z	((tx)y)z
t	e	tx	ty	tz	x	y	z	(tx)y	(tx)z	(ty)z	xy	xz	yz	((tx)y)z	(xy)z
x	-tx	-e	xy	xz	t	-(tx)y	-(tx)z	-y	-z	(xy)z	ty	tz	-((tx)y)z	-yz	(ty)z
y	-ty	-xy	-e	yz	(tx)y	t	-(ty)z	x	-(xy)z	-z	-tx	((tx)y)z	tz	xz	-(tx)z
z	-tz	-xz	-yz	-e	(tx)z	(ty)z	t	(xy)z	x	y	-((tx)y)z	-tx	-ty	-xy	(tx)y
tx	-x	-t	(tx)y	(tx)z	e	-xy	-xz	-ty	-tz	((tx)y)z	y	z	-(xy)z	-(ty)z	yz
ty	-y	-(tx)y	-t	(ty)z	xy	e	-yz	tx	-((tx)y)z	-tz	-x	(xy)z	z	(tx)z	-xz
tz	-z	-(tx)z	-(ty)z	-t	xz	yz	e	((tx)y)z	tx	ty	-(xy)z	-x	-y	-(tx)y	xy
xy	(tx)y	y	-x	(xy)z	ty	-tx	((tx)y)z	-e	yz	-xz	-t	(ty)z	-(tx)z	-z	-tz
xz	(tx)z	z	-(xy)z	-x	tz	-((tx)y)z	-tx	-yz	-e	xy	-(ty)z	-t	(tx)y	y	ty
yz	tyz	(xy)z	z	-y	((tx)y)z	tz	-ty	xz	-xy	-e	(tx)z	-(tx)y	-t	-x	-tx
(tx)y	xy	ty	-tx	((tx)y)z	y	-x	(xy)z	-t	(ty)z	-(tx)z	-e	yz	-xz	-tz	-z
(tx)z	xz	tz	-((tx)y)z	-tx	z	-(xy)z	-x	-(ty)z	-t	(tx)y	-yz	-e	xy	ty	y
(ty)z	yz	((tx)y)z	tz	-ty	(xy)z	z	-y	(tx)z	-(tx)y	-t	xz	-xy	-e	-tx	-x
(xy)z	-((tx)y)z	-yz	xz	-xy	(ty)z	-(tx)z	(tx)y	-z	y	-x	tz	-ty	tx	e	-t
((tx)y)z	-(xy)z	-(ty)z	(tx)z	-(tx)y	yz	-xz	xy	-tz	ty	-tx	z	-y	x	t	-e

Zero divisors

See: **Spacetime algebra zero divisor**.

Ideals

The STA has only two instead of four linearly independent minimal left ideals, because it has only half as many elements as the **Dirac algebra**.

An Observation

As the spacetime algebra $Cl(1,3)$ serves to describe spacetime, consequently $Cl(1,0)$ should allow for an adequate description of a world with **time** only (i.e. one with with space "stripped off"). In analogy, one could call this algebra the **Time Algebra**.

At first sight, doing physics with $Cl(1,0)$ may seem to be somewhat far-fetched. Yet it makes perfect sense, as the time algebra is a proper (even) subalgebra of the spacetime algebra and the latter therefore contains the "1-dimensional universe" described by $Cl(1,0)$ as a subspace.

$Cl(1,0)$ is isomorphic to the *complex numbers* and thus **geometric algebra** allows for an interesting reinterpretation of the complex numbers, namely as a 1-dimensional manifold (represented by tangent 1-blades) with a scalar field attached to it (0-blades).
In particular, the whole and extraordinarily rich apparatus of *complex analysis* applies to this Clifford algebra and well known theorems might find new geometric interpretations. E.g. the *Cauchy-Riemann equations* describe an intertwining of the manifold and the field on it.

In terms of physics this means that the very nature of time could be way more complicated that what one might naively think. Namely time inevitably comes along with a scalar field attached to it and there may be an intricate interplay between the two. In other words, the sheer mathematical depth of *complex analysis* should in principle also be part of a description of time.

Thus the following approach to **quantize gravity** is suggestive:
Start with a one-dimensional universe with time only and try to **quantize** it. In mathematical terms this means, find an adequate description of it by means of *discrete complex analysis*. (It is an intriguing fact, that two-dimensional gravity can be successfully quantized, although I do not know if this is relevant in this context ...).
(In the worst case this implies that one first has to proof the **Riemann hypothesis**, as it "sits" in the time algebra, before one can realize the "ultimate dream of physicists" to come up with a full fledged theory of quantum gravity, $:-)$).

The next step would be to include one space-dimension and try a quantization, i.e. to advance to $Cl(1,1)$.

Then one faces an interesting branching point. Including one more space-dimension, one can either do this by going to $Cl(1,2)$ or, e.g., to the **octonions**. (My guess is, that the octonions are the better choice ...).

Finally, including the third space-dimension, one has even more possibilities, depending on the doubling process one prefers. In the end comparing theoretical results with experiments may be the only way to find out which algebra is the correct one.

It is remarkable, that the relevance of an interpretation of time in terms of $Cl(1,0)$ and the resulting possibilities, just mentioned, seem to have been largely overlooked.

See also:

- **Clifford geometric algebra**
- *Lorentz Transformations* in Clifford spacetime
- *Electroweak theory and Clifford algebra*

Papers:

- A Unifying Clifford Algebra Formalism for Relativistic Fields (1983) - K. R. Greider local pct. 40
- Curvature Calculations with Spacetime Algebra (1986) - D. Hestenes local pct. 26
- Complex Geometry and Dirac Equation (1999) - S. De Leo, W. A. Rodrigues, Jr., J. Vaz, Jr. local pct. 14
- The Clifford Algebra of Space-time Applied to Field Theories, Part 1 (1987) P. G. Vroegindeweij local pct. 4
- Algebraic Structures on Space-time Algebra (1993) - P. G. Vroegindeweij local pct. 1
- [1] Basic Calculations on Clifford Algebras (2007) - G. Morales-Luna local pct. 0 - Contains a multiplication table of the STA and of $Cl(3,1)$.
- Algebraic Preliminaries for Field Theories in Space-time-algebra (1991) - P. G. Vroegindeweij local pct. 0

Documents:

- Spacetime Algebra in the Dirac Theory (1996) - P. G. Vroegindeweij local

Links:

- WIKIPEDIA - Spacetime Algebra
- Multiplication Table for Three Dimensional Clifford Algebra (ek2=+e0) - Multiplication table of $Cl(4)$. (Note, that the "untwisted" multiplication table of this algebra and the one of the spacetime algebra are **autotopic**).
- Homepage of Markus Maute - Clifford Multiplication Tables

Spreadsheets:

- Multiplication Table Spacetime Algebra (Works Format)

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Spacetime Algebra Zero Divisor

In the **spacetime algebra** one can distinguish three classes of primitive **idempotents** characterised by

$$\begin{aligned}\mathbf{D}_1 &= \frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_0)\\ \mathbf{D}_2 &= \frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{30}) \text{ and equivalently } \frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{10}) \text{ and } \frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{20})\\ \mathbf{D}_3 &= \frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{123})\end{aligned}$$

Theses involve all the basis elements that square like $\boldsymbol{\gamma}_{\dots}^2 = +\mathbf{1}$ but none of the kind $\boldsymbol{\gamma}_{\dots}^2 = -\mathbf{1}$. (See the multiplication tables under **spacetime algebra**).

Each of these idempotents can generate an equivalence class using the relations $\mathbf{D}_i' = \mathbf{G}\mathbf{D}_i\mathbf{G}^{-1}$ where, in this case, the $\mathbf{G}$ are elements of

the *spin group* **SL(2,ℂ)**.
The three defining idempotents can be related to each other through $\mathbf{D}_i = \mathbf{U}_{ij}\mathbf{D}_j\mathbf{U}_{ij}^{-1}$ where $\mathbf{U}_{ij}(\neq \mathbf{G})$ is some other invertible element of the algebra.

The reason for choosing one specific idempotent from the set is determined by the physics one is interested in. For example, the choice of $\frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_0)$ is made in order to pick out a particular Lorentz frame by fixing the **time** axis defined by $\boldsymbol{\gamma}_0$ through a *projection*. Any other Lorentz frame can be obtained by choosing the appropriate *Lorentz transformation* $\Lambda(\vec{v})$ from $SL(2,\mathbb{C})$ and forming $\boldsymbol{\gamma}_0' = \Lambda^{-1}(\vec{v})\boldsymbol{\gamma}_0\Lambda(\vec{v})$. In this way one relates the **algebra** to an equivalence class of Lorentz observers.

Alternatively $\frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{30})$ can be chosen if one wants to pick out a particular spin direction, the 3-axis in the Lorentz frame defined by $\boldsymbol{\gamma}_0$. In this case one is highlighting the even sub-algebra $Cl(1,3)^0 \cong Cl(3,0)$. Other spin directions can be described by transforming $\boldsymbol{\gamma}_{30}$ using appropriate elements of the spin group.

If one wants to consider the little group of the **Lorentz group**, $SO(2,1)$, then either $\frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{123})$ or $\frac{1}{2}(\mathbf{1}+\boldsymbol{\gamma}_{30})$ can be chosen. These elements project onto the idempotents of the corresponding **Clifford algebra** $Cl(2,1)$.

The three primitive idempotents themselves are equivalent under a larger **group**, the *Clifford group* which consists of all invertible elements of the algebra. For example we find the following relations

$$\begin{aligned}\mathbf{D}_2 &= \frac{1}{2}(\mathbf{1}-\boldsymbol{\gamma}_3)\mathbf{D}_1(\mathbf{1}+\boldsymbol{\gamma}_3)\\ \mathbf{D}_3 &= \frac{1}{2}(\mathbf{1}-\boldsymbol{\gamma}_5)\mathbf{D}_1(\mathbf{1}+\boldsymbol{\gamma}_5)\\ \mathbf{D}_3 &= \frac{1}{2}(\mathbf{1}-\boldsymbol{\gamma}_{012})\mathbf{D}_2(\mathbf{1}+\boldsymbol{\gamma}_{012})\end{aligned}$$

This group contains reflections and inversions and thus is related to **parity** and *time reversal*.

Papers:

- The Clifford Algebra Approach to Quantum Mechanics B: The Dirac Particle and its Relation to the Bohm Approach. (2010) - B. J. Hiley, R. E. Callaghan local pct. 7

Abstracts:

- Idempotent Structure of Clifford Algebras (1987) - P. Lounesto, G. P. Wene act. 25

Google books:

- Clifford Algebras and Spinor Structures (1995) - R. Abłamowicz, P. Lounesto local bct. 2 - P. 113-

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Split Algebra Zero Divisor

In the following we examine the characteristics of **zero divisors** in **algebras** having a split **signature**, that we will refer to as **Split Algebra Zero Divisors**.

Clifford algebras

For an example, see **spacetime algebra zero divisor**.

Cayley-Dickson algebras

The subsequent considerations apply to all split Cayley-Dickson algebras alike.

One can distinguish between two types of split algebra zero divisors (see also [1]), namely

Abelian zero divisors

These are ***projection operators*** with $\mathbf{D}^2 = \mathbf{D}$. (Note that every non-trivial **idempotent** is also a zero divisor).
We make the decompositions

$$\mathbf{e} = \frac{1}{2}(\mathbf{e}+\mathbf{e}_n) + \frac{1}{2}(\mathbf{e}-\mathbf{e}_n) \equiv \mathbf{D}_n^+ + \mathbf{D}_n^-$$

where $\mathbf{D}_n^+$ and $\mathbf{D}_n^-$ are assumed to be projection operators and $n \in \{1,\ldots,N-1\}$, with N the dimension of the algebra. Therefore

$$\mathbf{D}_n^{\pm}\mathbf{D}_n^{\pm} = \frac{1}{4}\left(\mathbf{e} \pm 2\mathbf{e}_n + \mathbf{e}_n^2\right)$$

such that we have to require $\mathbf{e}_n^2 = +\mathbf{e}$, which is the case for half of the N basis elements of a split algebra. (If only one "split-doubling step" is involved, these are exactly the basis elements containing the complex unit corresponding with this doubling step).
That is we get

$$\boxed{N_{\mathbf{D}^+} = N_{\mathbf{D}^-} = \frac{N}{2}}$$

projection operators, i.e. $N_{\mathbf{D}} \equiv N_{\mathbf{D}^+} + N_{\mathbf{D}^-} = N$ all in all.
Moreover

$$\mathbf{D}_n^+\mathbf{D}_n^- = \mathbf{D}_n^-\mathbf{D}_n^+ = \mathbf{0}$$

making them Abelian zero divisors.
(Alternatively one could use the notation $\mathbf{D}_n \equiv \mathbf{D}_n^+$ and $\mathbf{D}_n^* \equiv \mathbf{D}_n^-$, as the projection operators are related by complex conjugation).

Non-Abelian zero divisors

These are supposed to satisfy $\mathbf{G}^2 = \mathbf{0}$. (We'll refer to them as **Graßmann(-like)** operators as is done in [1]).
We make the decompositions

$$\mathbf{e}_l = \frac{1}{2}(\mathbf{e}_l + \mathbf{e}_m) + \frac{1}{2}(\mathbf{e}_l - \mathbf{e}_m) \equiv \mathbf{G}^+_{l,m} + \mathbf{G}^-_{l,m}$$

with $l, m \in \{1, \ldots, N-1\}$ and only consider $l \neq m$. As for CD algebras $\{\mathbf{e}_l, \mathbf{e}_m\} = \mathbf{0}$, to satisfy

$$\mathbf{G}^\pm_{l,m}\mathbf{G}^\pm_{l,m} = \frac{1}{4}\left(\mathbf{e}_l^2 + \mathbf{e}_m^2 \pm \{\mathbf{e}_l, \mathbf{e}_m\}\right) = \mathbf{0}$$

we have to require $\mathbf{e}_l^2 = -\mathbf{e}_m^2$, which is the case for

$$\boxed{N_{\mathbf{G}^+} = N_{\mathbf{G}^-} = \frac{N}{2}\left(\frac{N}{2} - 1\right)}$$

combinations of basis elements, which is determined by the signature $\left(\frac{N}{2}+1, \frac{N}{2}-1\right)$, such that we get $N_{\mathbf{G}} \equiv N_{\mathbf{G}^+} + N_{\mathbf{G}^-} = N\left(\frac{N}{2} - 1\right)$ Graßmann like zero divisors all in all.
(An aside: this construction does not work for **Clifford algebras** beyond the **quaternions** in the same manner as $\{\mathbf{e}_l, \mathbf{e}_m\} = 0$ with $n \neq m$ is not satisfied for all imaginary basis elements any more. One therefore expects less Graßmann like zero divisors for these algebras).

We do some redefinitions (making contact with the description in [1]). We assume that l is the index for which $\mathbf{e}_l^2 = +\mathbf{e}$ and m the one for which $\mathbf{e}_m^2 = -\mathbf{e}$. Then

$$\mathbf{G}^\pm_{l,m} = (\mathbf{e}_l \pm \mathbf{e}_m) = (\mathbf{e} \pm \mathbf{e}_n)\mathbf{e}_l = \mathbf{D}_n^\pm \mathbf{e}_l \equiv \mathbf{G}_n^\pm$$

It is important to realize that the map $\mathbf{G}^\pm_{l,m} = \mathbf{D}_n^\pm \mathbf{e}_l \to \mathbf{G}_n^\pm$ is unique (*"Latin square* property") whereas the map $\mathbf{D}_n^\pm \mathbf{e}_l \to \mathbf{G}_n^\pm$ is not. (Projection versus decomposition).
In fact, given a $\mathbf{G}_n^\pm$, we can assign a $\mathbf{D}_n^\pm$ in $M = \frac{N}{2} - 1$ different ways (the number of elements the index l is running over), where M is determined by the signature $(N - M, M)$ of the algebra.

Interestingly, an inspection of some multiplication tables suggests that for those columns where the basis elements square to $-\mathbf{e}$ the number of negative and positive signs of the *structure constants* is equal, which by no means is so in general for split CD algebras. (Unfortunately I do not have a proof for that, but I guess it should be possible to show it by induction with the **Cayley-Dickson doubling formula**).

Thus for a given m we get as many projection operators as Graßmann operators, justifying the usage of the same index n for both of them.

Examples

Up to the **octonions** all split Cayley-Dickson algebras are **isomorphic** which allows for picking one of them pars pro toto.

Split complex numbers

$$\mathbf{D}^\pm = \frac{1}{2}(\mathbf{e} \pm \mathbf{i})$$

As this algebra is commutative there are no Graßmann like zero divisors. Thus the number of zero divisors coincides with the number of idempotents in this algebra.

Split quaternions

We use the algebra $CD(1, -1)$, having the following **multiplication table**

$\mathbf{e}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\mathbf{e}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{e}$	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$\mathbf{e}$

The projectors are

$$\begin{aligned}\mathbf{D}_\mathbf{j}^\pm &= \frac{1}{2}(\mathbf{e} \pm \mathbf{j})\\ \mathbf{D}_\mathbf{k}^\pm &= \frac{1}{2}(\mathbf{e} \pm \mathbf{k})\end{aligned}$$

and the Graßmann operators

$$\begin{aligned}\mathbf{G}_{\mathbf{i},\mathbf{j}}^\pm &= \frac{1}{2}(\mathbf{i} \pm \mathbf{j})\\ \mathbf{G}_{\mathbf{i},\mathbf{k}}^\pm &= \frac{1}{2}(\mathbf{i} \pm \mathbf{k})\end{aligned}$$

Alternatively, these can be writen as

$$\begin{aligned}\frac{1}{2}(\mathbf{i} \pm \mathbf{j}) &= \frac{1}{2}(\mathbf{e} \pm \mathbf{k})\mathbf{i} \equiv \mathbf{G}_\mathbf{k}^\pm\\ \frac{1}{2}(\mathbf{i} \pm \mathbf{k}) &= \frac{1}{2}((\mathbf{e} \mp \mathbf{j})\mathbf{i} \equiv \mathbf{G}_\mathbf{j}^\mp\end{aligned}$$

We just mention that the structure constants relating the two notations are given by the *Levi-Civita tensor*. In case of the quaternions the map between the two representations is $1:1$.

Split Octonions

Split Sedenions

Papers:

- Exceptional Projective Geometries and Internal Symmetries (2003) - S. Catto local pct. 23
- [1] Observable Algebra (2003) - M. Gogberashvili local pct. 2
- Left-Ideals, Dirac Fermions and SU(2)-Flavour (2007) - F. M. C. Witte local pct. 0

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Split Quaternion

There are six variants of ***Split* Quaternion Algebras**, three "right-handed" ones, $CD(1,-1)$, $CD(-1,1)$ and $CD(-1,-1)$ obtained by the classical **Cayley-Dickson doubling process**, and three "left-handed" ones. For their multiplication tables, see **quaternion multiplication tables**.

The "right-handed" $CD(1,-1)$-algebra is also known as the algebra of **coquaternions (para-quaternions)**.

Papers:

- Quaternionic and Poisson-Lie structures in 3d Gravity: the Cosmological Constant as Deformation Parameter (2007) - C. Meusburger, B. J. Schroers local pct. 10

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Split-Biquaternion

The algebra of **Split-Biquaternions** $\mathbb{H}_{\tilde{\mathbb{C}}}$ is defined by $H_{\tilde{\mathbb{C}}} \equiv \tilde{\mathbb{C}} \otimes \mathbb{H}$ where $\tilde{\mathbb{C}}$ and $\mathbb{H}$ are the algebras of **split complex numbers** and **quaternions** respectively.

See also:

- **Biquaternion**

Links:

- WIKIPEDIA - Split-biquaternion

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Subalgebra

A **Subalgebra** of an **algebra** is a subset which is closed under all the operations of the algebra.

A moderate sized algebra may have an enormous number of subalgebras. One therefore introduces the concept of a **Maximal Subalgebra**. It is a subalgebra which is not containted in any larger subalgebra.
A maximal subalgebra of a *semi-simple* algebra need not be semi-simple.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Superalgebra

An **algebra** $\mathcal{A}$ is called a **Superalgebra** if it is $\mathbb{Z}_2$-**graded**, i.e. there exist linear subspaces $\mathcal{A}_i$ with $i \in \{0,1\}$ such that

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$$

and

$$\mathcal{A}_0\mathcal{A}_0 \subseteq \mathcal{A}_0, \quad \mathcal{A}_0\mathcal{A}_1 \subseteq \mathcal{A}_1, \quad \mathcal{A}_1\mathcal{A}_0 \subseteq \mathcal{A}_1, \quad \mathcal{A}_1\mathcal{A}_1 \subseteq \mathcal{A}_0$$

$\mathcal{A}_0$ is called **even** and $\mathcal{A}_1$ **odd** part of $\mathcal{A}$.
Notice that a superalgebra is neither supposed to be commutative nor associative.

Examples

- **Lie superalgebras**
- **Clifford algebras**
- **Grassmann algebras**
- Malcev superalgebras

Papers:

- Maximal Subalgebras of Simple Alternative Superalgebras (2009) - J. Laliena, S. Sacristan local pct. 0
- Akivis Superalgebras and Speciality (2007) - H. Albuquerque, A. P. Santana local pct. 0

Links:

- WIKIPEDIA - Superalgebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Superconformal Algebra

Papers:

- Properties of Conformal Supergravity (1978) - M. Kaku, P. K. Townsend, P. Van Nieuwenhuizen local pct. 205

Links:

- WIKIPEDIA - Superconformal Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Tangent Algebra

> Quantities which describe how the state of a spatially extended physical system varies from point to point have not a distinct value but only one "for each point": in mathematical language they are "functions of the place or point". According as we are dealing with a scalar, vector, or tensor, we speak of a scalar, vector, or tensor field.
> - Hermann Weyl - Space-Time-Matter -

This is a ***Draft*** *!*

As a "warm up" let's regard a 1-dimensional manifold, given by a classical scalar function $\Phi(x)$ in one variable.
Using the series expansion of the exponential function one can write

$$\begin{aligned}\exp(dx\partial_x)\Phi(x) &= \Phi(x+dx)\\ &= \Phi(x) + dx\partial_x\Phi(x) + \frac{1}{2}\,dx^2\partial_x^2\Phi(x) + \frac{1}{3!}\,dx^3\partial_x^3\Phi(x) + \dots\end{aligned}$$

The same result is obtained by doing a Taylor series expansion of $\Phi(x)$.

Alternatively this can be expressed in tensorial form as follows

$$\boldsymbol{d}\Phi(x) \equiv c_1(x)\boldsymbol{dx} + \frac{1}{2}\,c_2(x)\boldsymbol{dx}\otimes\boldsymbol{dx} + \frac{1}{3!}\,c_3(x)\boldsymbol{dx}\otimes\boldsymbol{dx}\otimes\boldsymbol{dx} + \dots$$

with $c_i(x) \equiv \partial^i\Phi(x)$ defining (local) coefficients, corresponding to the i-th order of the expansion.
$\boldsymbol{d}\Phi$ is known as **total differential** of Φ.
The "dual" and equivalent description is given by

$$\boldsymbol{d}\Phi(x) \equiv \tilde{c}_1(x)\boldsymbol{\partial}_x\Phi(x) + \frac{1}{2}\,\tilde{c}_2(x)\boldsymbol{\partial}_x\otimes\boldsymbol{\partial}_x\Phi(x) + \frac{1}{3!}\,\tilde{c}_3(x)\boldsymbol{\partial}_x\otimes\boldsymbol{\partial}_x\otimes\boldsymbol{\partial}_x\Phi(x) + \dots$$

with the coefficients $\tilde{c}_i$ now defined by powers of infinitesimal changes dx. (One might consider absorbing Φ in the coefficients).

For the 2-dimensional case we get

$$\begin{aligned}\exp(dx\partial_x + dy\partial_y)\Phi(x,y) &= \Phi(x+dx, y+dy)\\ &= \Phi(x,y) + dx\partial_x\Phi(x,y) + dy\partial_y\Phi(x,y) + \frac{1}{2}\,dx^2\partial_x^2\Phi(x,y)dx^2 + dxdy\partial_x\partial_y\Phi(x,y) + \frac{1}{2}\,dy^2\partial_y^2\Phi(x,y) +\\ &\quad \frac{1}{3!}\,dx^3\partial_x^3\Phi(x,y) + \frac{1}{2}\,dx^2dy\partial_x^2\partial_y\Phi(x,y) + \frac{1}{2}\,dxdy^2\partial_x\partial_y^2\Phi(x,y) + \frac{1}{3!}\,dy^3\partial_y^3\Phi(x,y) + \ldots\end{aligned}$$

As the order of the sum in the argument of the exponential function doesn't matter, so doesn't the order of the differentiations in x and y in the expansion.

In the n-dimensional case the total differential reads

$$\begin{aligned}\boldsymbol{d}_{\mathbf{x}}\Phi(\mathbf{x}) &= \exp(dx^\mu\partial_{x^\mu})\Phi(\mathbf{x})\\ &= \sum_{k=1}^{\infty}\frac{1}{k!}\,(dx^\mu\partial_{x^\mu})^k\Phi(\mathbf{x})\\ &= \sum_{\mu_1=1}^{\infty}\cdots\sum_{\mu_n=1}^{\infty}\frac{dx^{\mu_1}\ldots dx^{\mu_n}}{\mu_1!\ldots\mu_n!}\,\partial_{x_1}\ldots\partial_{x_n}\Phi(\mathbf{x})\end{aligned}$$

with $\mu = 1,\ldots,n$ and $\mathbf{x} = (x^1,\ldots,x^n)$.

Applying the $\exp$-operator a second time involves an algebraic product, i.e. we must specify how to multiply two $\exp$-functions. E.g. if the product is non-commutative, the order matters. If applying the $\exp$-operator more than 2 times one furthermore has to consider the issue of non-associatvity of the product.
To stay general, we'll assume a non-associative product. I.e. the related n-dimensional manifold is supposed to be a **quasigroup manifold**.

Given a differentiable **quasigroup** $\mathcal{Q}$ (we'll restrict ourselves to **binary quasigroups** in the following) and elements $\mathbf{A},\mathbf{B}\in\mathcal{Q}$ (infinitesimally) close to the identity $\mathbf{e}$, their (quasigroup-)product can be written as

$$\begin{aligned}\mathbf{AB}\equiv\mathbf{C} &= \exp(\boldsymbol{d}a^\mu\partial_{a^\mu})\exp(\boldsymbol{d}b^\mu\partial_{b^\mu})\Phi(\mathbf{x})\\ &\equiv \exp(\boldsymbol{D}c^\mu\partial_{c^\mu})\Phi(\mathbf{x})\\ &= \exp(\boldsymbol{D}c^\mu(a^1,\ldots a^n,b^1,\ldots b^n)\partial_{c^\mu})\Phi(\mathbf{x})\\ &= \exp(\boldsymbol{D}c^\mu(\mathbf{a},\mathbf{b})\partial_{c^\mu})\Phi(\mathbf{x})\end{aligned}$$

with $d^\mu(\mathbf{a},\mathbf{b})$ functions that are dependent on the algebra.

So geometrically speaking, given a manifold and doing an infinitesimal transformation from a point $\mathbf{O}$ in the origin (algebraically the identity) to a point $\mathbf{A}$, followed by an infinitesimal transformation from $\mathbf{A}$ to $\mathbf{C}$ by $\mathbf{B}$ results in a point with a location depending on the algebra describing the manifold.

The formula above is the **Baker-Campbell-Hausdorff formula** in a very general form. (Hence, if the quasigroup is a group one has to recover the classical formula).
It is a map of the quasigroup product to an (Abelian) sum in the argument of the exponential. Properties of the product like non-commutativity or non-associativity are now coded in the coefficients of the terms. The "price one has to pay" is that these coefficients may be quite complicated. On the other hand, familiar entities like the **torsion tensor** or the **Riemann tensor** will show up, i.e. the abstract quasigroup product is transformed to a form which might be better interpretable. This is relevant in respect to potential applications of quasigroup manifolds to physics.
As quasigroups have tangent algebras of higher orders than have **Lie groups**, quasigroup manifolds are capable of coding more information than are *Lie group manifolds*. Therefore one might speculate if 4 dimensions could do to code all of known physics, making extra dimensions superfluous (for which experimental evidence seems to be lacking).

As is the case for Lie theory, locally one has a dual description, either in the terms of the quasigroup or in terms of the tangent algebra. In fact there exists a homomorphism between the two, given by the *exponential map*. Depending one the concrete situation, the one or the other point of view might be advantageous.

One can interpret $\exp(\boldsymbol{d}x^i\partial_{x^i}) = \exp(dx^i\mathbf{e}_i)$ as a "generalized translation"-operator, as for Abelian groups in fact one gets translations. The tangent space in this case is a classical vector space and the group product is transformed to vector addition in the tangent space which corresponds with translations.
In the more general setting of a quasigroup product, the $\exp$-operation can be interpreted as describing a very generic form of "parallel transport" in the manifold, including classical parallel transport. If expressed by means of a **geodesic product**, the geometric interpretation is that of parallel transport of (infinitesimal) line segments in the manifold.

Next we do a **canonical expansion** of the $c^\mu(\mathbf{a},\mathbf{b})$:

$$\begin{aligned}\boldsymbol{D}c^\mu(\mathbf{a},\mathbf{b}) &= \frac{\partial c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu}\,\boldsymbol{d}a^\nu + \frac{\partial c^\mu(\mathbf{a},\mathbf{b})}{\partial b^\nu}\,\boldsymbol{d}b^\nu + \frac{\partial^2 c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu\partial b^\rho}\,\boldsymbol{d}a^\nu\otimes\boldsymbol{d}b^\rho +\\ &\quad \frac{1}{2}\left(\frac{\partial^3 c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu\partial a^\rho\partial b^\sigma}\,\boldsymbol{d}a^\nu\otimes\boldsymbol{d}a^\rho\otimes\boldsymbol{d}b^\sigma + \frac{\partial^3 c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu\partial b^\rho\partial b^\sigma}\,\boldsymbol{d}a^\nu\otimes\boldsymbol{d}b^\rho\otimes\boldsymbol{d}b^\sigma\right) +\\ &\quad \frac{1}{3!}\,\frac{\partial^4 c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu\partial a^\rho\partial a^\sigma\partial b^\tau}\,\boldsymbol{d}a^\nu\otimes\boldsymbol{d}a^\rho\otimes\boldsymbol{d}a^\sigma\otimes\boldsymbol{d}b^\tau + \frac{1}{2\cdot 2}\,\frac{\partial^4 c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu\partial a^\rho\partial b^\sigma\partial b^\tau}\,\boldsymbol{d}a^\nu\otimes\boldsymbol{d}a^\rho\otimes\boldsymbol{d}b^\sigma\otimes\boldsymbol{d}b^\tau +\\ &\quad \frac{1}{3!}\,\frac{\partial^4 c^\mu(\mathbf{a},\mathbf{b})}{\partial a^\nu\partial b^\rho\partial b^\sigma\partial b^\tau}\,\boldsymbol{d}a^\nu\otimes\boldsymbol{d}b^\rho\otimes\boldsymbol{d}b^\sigma\otimes\boldsymbol{d}b^\tau + \sum_{g=5}^{\infty}\mathfrak{p}_g(\{\mathbf{da},\mathbf{db}\})\\ &\equiv \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\nu}\,c^\mu(\mathbf{a},\mathbf{b})db^\nu + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\rho}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu db^\rho +\\ &\quad \frac{1}{2}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\rho}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\sigma}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu da^\rho db^\sigma + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\rho}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\sigma}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu db^\rho db^\sigma\right) +\\ &\quad \frac{1}{3!}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\rho}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\sigma}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\tau}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu da^\rho da^\sigma db^\tau + \frac{1}{2\cdot 2}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\rho}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\sigma}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\tau}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu da^\rho db^\sigma db^\tau +\\ &\quad \frac{1}{3!}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}a^\nu}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\rho}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\sigma}\otimes\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}b^\tau}\,c^\mu(\mathbf{a},\mathbf{b})da^\nu db^\rho db^\sigma db^\tau + \sum_{g=5}^{\infty}\mathfrak{p}_g(\{\mathbf{da},\mathbf{db}\})\end{aligned}$$

In the second expression the basis is represented by means of elements of the underlying vector space, whereas in the first expression it is represented by means of elements of its dual space. (Both representations are equivalent).

The coefficients of the monomials are the partial derivatives of the scalar functions $c^\mu(\mathbf{a},\mathbf{b})$ with respect to the variables a_μ and b_ν, and this is the reason that these coefficients are not **tensors**. (As the *affine connection* $\Gamma_{\mu\nu}$ occurs as part of the second order of this expansion, this also explains the well known fact that its coefficients are in general not tensorial).

It seems that the representation of the basis in terms of forms bears some advantages, e.g. in respect to integration theory. In this case one decouples the operation of differentiation and the algebraic product. (Having an object with two actions may be more quite difficult to handle and confusing).
As both representations are equivalent, one trick we'll use is to switch back and forth "on the fly" between the two representation, depending on which one is better suited in a given situation.

Note, that only 3 out of the 5 possible **association types** of degree 4 show up. I.e. the expansion above is not describing the most general manifold possible. This shortcoming may be "cured" by considering **n-quasigroups** instead of binary ones, which can be done within the framework of **n-web theory**. When it comes to a more general description of spacetime, this suggests to also consider ternary quasigroups and 4-web theory.

A word about notation

The expression above shows the duality between the "$\boldsymbol{d}$'s" and "$\boldsymbol{\partial}$'s", i.e. the space of forms and the space of differential operators and we can equally well identify either one with the basis elements. Depending on which way around this is done, the symbols representing the basis are marked boldface, whereas the "dual" symbols, representing the coefficients, are not. In case that the basis is given by differential operators, the coefficients correspond with n-form fields.

$$\begin{array}{ccccc} da^{\mu\nu\cdots} & \mathbf{e}_\mu \otimes \mathbf{e}_\nu \otimes \ldots & \leftrightarrow & \boldsymbol{da}^\mu \otimes \boldsymbol{da}^\nu \otimes \ldots & e_\mu e_\nu \ldots \\ \text{coordinates} & \text{basis} & \text{dual} & \text{basis} & \text{coordinates} \end{array}$$

As Φ is supposed to contain all the information needed to reconstruct the manifold, we will call it the generating function of the manifold. In case of a 4-dimensional spacetime manifold we'll name it **World Function**. (The idea is that this function codes all physical information of the "world", spacetime and matter alike. One might even go as far as to interpret Φ as a probability function. (Which is suggestive as it is a scalar function). Considering a relationship with the Hartle-Hawking **wavefunction of the universe** would then be an option too. (At any rate we would have a generalisation of the latter, as we are in a more general, i.e. nonassociative, setting and an equation as "simple" as the **Wheeler-deWitt equation**, which is semiclassical anyway, will no do any more. Maybe this way one could get rid of problems associated with this function).

Note, that in the general case of a quasigroup manifold we cannot - as is the case for Lie groups - reconstruct the manifold uniquely from local (differential) information. (The **converse Lie theorem** does not generally apply in this situation). That is to say that a tangent algebra is not sufficient for the description of the manifold. We therefore need to know the function Φ globally. In terms of physics this implies, that the extraction of local information is not sufficient for making precise predictions. In fact we would have to determine the function Φ in every point in the manifold, which is at least practically impossible.

Some further thoughts and speculations

The following relationships are suggestive, yet they require a further elaboration and a better understanding (.... unfortunately time is quite of an enemy when it comes to doing fundamental physics ...):

It seems that the tangent algebra description given above can be cast in the form of a **spectral triple**. At least, we have all ingredients required:
An algebra $\mathcal{A}$ (given by the algebraic product between the exponential functions).
A **Dirac operator**, which is however a bit of a subtle issue, as it depends on what one understands by that notion. At any rate, the description here considerably generalises the concept of a classical Dirac operator, as higher order derivatives are involved, yet it should be possible to recover the classical Dirac operator when suppressing these.
Finally Ψ can be taken as an element of a **Hilbert space**. Yet this only applies in the "lower order situation", mentioned. Yet, in the more general case this does not imply that Φ is totally "crazy" and ill behaved. Instead, first of all it should be a C^4-function. Moreover it is reasonable to presuppose that it is (square) integrable, otherwise we cannot interpret it as a probability function and give it a physical meaning. Therefore if constraining the tangent space appropriately, it is conceivable that one can recover the *noncommutative standard model*, yet it "sits" within a larger (nonassociative) geometrical structure, potentially allowing one to go beyond it. Consequently, the Hilbert space fails when taking into account higher orders of the tangent space (in principle from order 3 onwards). I.e. a nonlinear description takes over and replaces the "classical" quantum description. This raises hopes that the additional (nonlinear) terms render the theory **renormalizable**.

Papers:

- Lie's fundamental Theorems for Local Analytical Loops - K. H. Hofmann, K. Strambach local pct. 13
- Smooth Quasigroups and Geometry - P. O. Mikheev and L. V. Sabinin local pct. 7 prl. 10
- Moufang symmetry I. Generalized Lie and Maurer-Cartan Equations - E. Paal pct. 5 prl. 9
- Deformations of Ternary Algebras - H. Ataguema, A. Makhlouf local pct. 4
- Local Algebras of a Differential Quasigroup - M. A. Akivis, V. V. Goldberg local pct. 2 prl. 10
- Smooth Quasigroups and Loops: Forty five Years of Incredible Growth - Lev V. Sabinin pct. 2
- Analog of Lie Algebra and Lie Group for Quantum Non-Hamiltonian Systems (1996) - V. E.Tarasov pct. 0
- On the Tangent Algebra of Algebraic Commutative Moufang Loops - G. P. Nagy pct. 0
- Classification of Multidimensional Three-Webs by Closure Conditions - A. M. Shelekhov
- A Note on the Akivis Algebra of a Smooth Hyporeductive Loop - A. N. Issa local pct. 0 prl. 5

Links:

- WIKIPEDIA - Taylor Series
- A Seminar in Group Theory, with Extra Advanced Topics - D. Finley

Google books:

- Geometry and Algebra of Multidimensional Three-webs (1992) - M. A. Akivis, A. M. Shelekhov local bct. 64 brl. 10
- Computational Commutative and Non-commutative Algebraic Geometry - S. Cojocaru, G. Pfister and V. Ufnarovski bct. 9 - about the non-commutative Taylor series.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Temperley-Lieb Algebra

The **Temperley-Lieb Algebra** is related to subfactors of **Von Neumann algebras**.

Links:

- WIKIPEDIA - Temperley-Lieb Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Ternary Algebra

See also:

- **N-Webs**
- **N-quasigroups**
- **Comtrans algebra**

Papers:

- On a Class of Ternary Composition Algebras (1996) - A. Elduque local pct. 10
- Ternutator Identities (2009) - C. Devchand, D. Fairlie, J. Nuyts, G. Weingart local pct. 8
- Ternary Numbers and Algebras Reflexive Numbers and Berger Graphs (2007) - A. Dubrovski, G. Volkov local pct. 4
- Necessary Conditions for Ternary Algebras (2010) - D. B. Fairlie, J. Nuyts local pct. 2

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Twisted Cayley-Dickson Algebras

In [1] D. Chesley describes what he calls **Twisted Cayley-Dickson Algebras**, using either one of the following two modified **Cayley-Dickson doubling** formulas

$$\begin{aligned} (\mathbf{A}_1, \mathbf{A}_2)(\mathbf{B}_1, \mathbf{B}_2) &= (\mathbf{A}_1\mathbf{B}_1 - \mathbf{A}_2^*\mathbf{B}_2, \mathbf{B}_2\mathbf{A}_1^* + \mathbf{A}_2\mathbf{B}_1) \\ (\mathbf{A}_1, \mathbf{A}_2)(\mathbf{B}_1, \mathbf{B}_2) &= (\mathbf{A}_1\mathbf{B}_1 - \mathbf{A}_2\mathbf{B}_2^*, \mathbf{A}_1\mathbf{B}_2 + \mathbf{B}_1^*\mathbf{A}_2) \end{aligned}$$

which are equivalent in the same way as are the two variants of the classical Cayley-Dickson doubling formulas.

These formulas contain the same elements as the classical Cayley Dickson formulas, however they differ from the latter in that $\mathbf{A}_2$ and $\mathbf{B}_2$ are exchanged in the first component of the doubled algebra and conjugation is exchanged between $\mathbf{A}_1$ and $\mathbf{B}_1$ in the second component.

They are the only alternatives to the classical doubling formulas if one requires

- $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$
- and the existence of 7 quaternionic triads

for the doubled algebra in case of dimension 8.

The relevance of twisted CD algebras is given at least by the fact, that twisted octonions naturally occur as **subalgebras of (untwisted) classical Cayley Dickson sedenion algebras**.

The differences between twisted Cayley-Dickson algebras and classical Cayley-Dickson algebras are solely due to their associated **sign tables**, their underlying **Fano spaces** are the same.

See also: **twisted octonions**.

Papers:

- [1] Superparity and Curvature of Twisted Octonionic Manifolds Embedded in Higher Dimensional Spaces - D. Chesley pct. 0 prl. 10

Links:

- Detailed Derivation and Generalization of Cayley-Dickson Construction - D. Chesley

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Universal Algebra

Universal Algebra studies **algebraic structures** abstractly, rather than specific types of structures as is done in **abstract algebra**.

Links:

- WIKIPEDIA - Universal Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Universal Enveloping Algebra

The construction of the **Universal Enveloping Algebra** of a **Lie algebra** is useful in order to pass from a non-associative structure to a more familiar (associative) algebra over the same field while preserving the representation theory.

Example

The **Weyl algebra** is the universal enveloping algebra of the **Heisenberg algebra**.

Papers:

- An Envelope for Malcev Algebras (2004) - J. M. Pérez-Izquierdo, I. P. Shestakov local pct. 47
- An Envelope for Bol Algebras (2005) - J. M. Pérez-Izquierdo local pct. 17
- Universal Enveloping Algebras and Some Applications in Physics (2005) - X. Bekaert local pct. 3

Presentations:

- Enveloping Algebra for Simple Malcev Algebras (2010) E. Barreiro local

Links:

- WIKIPEDIA - Universal Enveloping Algbebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Variational Principle for Algebras

Usually the *principle of least action* is applied to an action which is built from the physical fields in regards. Yet the algebraic structure underlying the fermionic part of the world stays fixed (e.g. the Dirac algebra or the **signature** of spacetime). The question is whether there is a deeper reason for it to be the way it is. (See also **dimensionality of the world**).

An **algebra** (at least a finite dimensional one) can be defined by its *structure constants*. It is therefore suggestive to treat these in the same manner as one does conventional physical fields and subject them to a variation. (What is more, IF one interprets the **automorphisms** between different multiplication tables as gauge fields which in fact are varied in the usual physical situation, it seems only consequent to also vary the structure of an algebra representing an "orthonormal" table, corresponding with the very nature of fermionic fields).

So, roughly speaking, one could extend the classical variational principle by considering:

- Values of the signs of the structure constants (i.e. one varies the **(co)homology** or in other words the twisting of the **cochain**). A variation of the signature would be a special case thereof (e.g. see [1]). This manœuvre can also be understood as a variation of a generalized **parity**.
- A "reshuffling" of the elements of the multiplication table. This corresponds with **autotopisms** of the algebra. (Touching the integrity of the underlying Latin square of the multiplication table on the other hand seems to be less favourable). The physical meaning of autotopisms yet remains to be understood.
- The dimension of the algebra.
- **n-ary algebras** and vary the parameter n.

Papers:

- [1] Duality and Strings, Space and Time (1999) - C. M. Hull local pct. 26

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Vertex Operator Algebra

The mathematical notion of a **Vertex Algebra** was introduced by Richard Borcherds. A variant of it, called **Vertex Operator Algebra**, was introduced by Igor Frenkel, James Lepowsky and Arne Meurman.
These notions are algebraic formulations of concepts that had been developed by many string theorists, conformal field theorists and quantum field theorists, and formalized as certain "operator algebras", later called **Chiral Algebras** in physics.

Papers:

- Some Developments in Vertex Operator Algebra Theory, Old and New - J. Lepowsky pct. 2

Links:

- The Online Database of Vertex Operator Algebras and Modular Categories - T. Gannon, G. Höhn, H. Yamauchi

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Virasoro Algebra

A **Virasoro Algebra** is an example of an infinite-dimensional **Lie algebra**.

It is a complex Lie algebra spanned by L_n, $n \in \mathbb{Z}$, and the **Central Element** C with **Lie bracket**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\,\delta_{m+n,0}\,C$$

where $\delta_{k,0} = 1$ if $k = 0$ and $\delta_{k,0} = 0$ otherwise.

A representation of a Virasoro algebra is generated by a highest **weight vector** v of a **Verma Module** $M(c,h)$ with

$$Cv = c$$
$$L_0 v = hv,\;\; L_n v = 0$$

with $n \geq 1$ and $c, h \in \mathbb{R}$.
c is called **central charge** and h **Conformal Weight** of the module.

A basis of $M(c,h)$ is given by

$$\{L_{-m_1} \cdots L_{-m_k} v \mid k, m_1, \ldots, m_k \in \mathbb{Z},\, m_1 \geq \cdots \geq m_k \geq 1\}$$

It can be shown that $M(c,0)$ is a **simple vertex operator algebra**.

The role of the Virasoro algebra in mathematical physics is that it describes the infinitesimal symmetries of a closed circle, in particular a closed string in **string theory**.

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

Von Neumann Algebra

A **Von Neumann Algebra** or **W*-algebra** is a ***-algebra** of *bounded operators* on a **Hilbert space** that is closed in the *weak operator topology* and contains the identity operator.

Since the weak topology is coarser than the uniform topology, every von Neumann algebra is a **C*-algebra**, whence all the result about C*-algebras also hold for von Neumann algebras.

Theses:

- Types of von Neumann Algebras (2011) - E. L. Jacobs local

Videos:

- Factors and Geometry (2001) - A. Connes

Books:

- Operator Algebras Theory of C*-Algebras and von Neumann Algebras (2013) - B. Blackadar local bct. 204

Links:

- WIKIPEDIA - Von Neumann Algebra

Your comments are very much appreciated. Suggestions, questions, critique, ... ?

X-Product

*This is a **Draft** !*

Given $\mathbf{X} \in \mathbb{O}$ with unit norm, the **(Octonion) X-Product** $\circ_{\mathbf{X}}$ is defined by the map $\circ_{\mathbf{X}} : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$,

$$\mathbf{A} \circ_{\mathbf{X}} \mathbf{B} \equiv (\mathbf{A}\mathbf{X})(\mathbf{X}^{\dagger}\mathbf{B}) = \mathbf{X}((\mathbf{X}^{\dagger}\mathbf{A})\mathbf{B}) = (\mathbf{A}(\mathbf{B}\mathbf{X}))\mathbf{X}^{\dagger}$$

for any $\mathbf{A}, \mathbf{B} \in \mathbb{O}$. (E.g. see [1]).
For $\mathbf{X}$ fixed, the algebra defined by $\mathbb{O}$ with the X-product is isomorphic to $\mathbb{O}$ with the conventional product. The X-product then is a **principal orthogonal autotopy** (with $\mathbf{A} \circ_{\alpha,\beta,\gamma} \mathbf{B} \equiv \mathbf{A} \circ_{\varphi,\varphi^{-1},1} \mathbf{B}$ and $\varphi : \mathbb{O} \mapsto \mathbb{O}$ given by the octonionic product). In terms of the **multiplication tables** this means that the border elements of the columns and rows are permuted in a related way. The "inner" elements of the table stay the same. I.e. the *projective geometry* of the octonions, given by a **Fano plane**, is not affected.

The X-product is a special case of the **XY-Product** which is defined by

$$\mathbf{A} \circ_{\mathbf{X},\mathbf{Y}} \mathbf{B} \equiv (\mathbf{A}\mathbf{X})(\mathbf{Y}^{\dagger}\mathbf{B})$$

which is also a principal orthogonal autotopy (with $\mathbf{A} \circ_{\alpha,\beta,\gamma} \mathbf{B} \equiv \mathbf{A} \circ_{\varphi,\varphi',1} \mathbf{B}$ and φ, φ' being **orthogonal maps**). In terms of the multiplication tables this means that the border elements of the columns and rows are permuted in an unrelated way. The "inner" elements of the table stay the same.

The X-Product is interesting in regards to its action on the *480 non-equivalent octonion multiplication tables*: For a discrete set Ξ of 480 vectors $\mathbf{X}_i$, consisting of the union of 2 copies of the set of **root vectors** of the **Lie algebra E_8** (spanning a pair of **E_8-lattices**), the map

$\gamma : \mathbf{X}_i \mapsto \mathbf{X}_j$ relates different multiplication tables, leaving the identity fixed. In a similar way, when considering the set of roots of the **Λ_{16}-lattice**, the XY-products relates the $7.680 = 2 \cdot 3.840$ different multiplication tables that arise when not requiring that the identity be fixed. It has $4.320 = 3.840 + 480$ roots and the XY-product acts on 2 sets of the Λ_{16} roots. (For details see [2]).

E_8 roots

For the set $\Xi = \Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \Xi_3$ of root vectors with

$$\begin{aligned}
\Xi_0 &= \left\{ \frac{1}{\sqrt{1}} (\pm \mathbf{e}_a) \right\} \\
\Xi_1 &= \left\{ \frac{1}{\sqrt{2}} (\pm \mathbf{e}_a \pm \mathbf{e}_b) \right\} \\
\Xi_2 &= \left\{ \frac{1}{\sqrt{4}} (\pm \mathbf{e}_a \pm \mathbf{e}_b \pm \mathbf{e}_c \pm \mathbf{e}_d) \right\}, \ \mathbf{e}_a(\mathbf{e}_b(\mathbf{e}_c\mathbf{e}_d)) = \pm 1 \\
\Xi_3 &= \left\{ \frac{1}{\sqrt{8}} (\sum_{a=0}^{7} \pm \mathbf{e}_a) \right\}, \ \text{number of } +\text{'s odd}
\end{aligned}$$

and $a, b, c, d, e, f, g \in \{0, \ldots, 7\}$ distinct in each case $i = 1, 2, 3$, one gets

$$\mathbf{e}_a \circ_{\mathbf{X}} \mathbf{e}_b = \tilde{\mathbf{e}}_c$$

with $\mathbf{X} \in \Xi$, where "$\circ_{\mathbf{X}}$" mimics the action of the product of one of the 480 octonion multiplication tables. I.e. in effect "$\circ_{\mathbf{X}}$" is different from the original product, defined by $\mathbf{e}_a \cdot \mathbf{e}_b = \mathbf{e}_c$, although "$\circ_{\mathbf{X}}$" is based on "$\cdot$". Only if $\mathbf{X} = \mathbf{e}$ or $\mathbf{X} = -\mathbf{e}$ do the products coincide, i.e. $\mathbf{e}_c = \tilde{\mathbf{e}}_c$. Note however, that the subsets Ξ_i are not general. They at least work for certain original products "$\cdot$".

For the **orders** of the subsets one has

$$\begin{aligned}
\text{ord}(\Xi_0) &= 2 \cdot 8 = 16 \\
\text{ord}(\Xi_1) &= 4 \cdot 28 = 112 \\
\text{ord}(\Xi_2) &= 2 \cdot 112 = 224 \\
\text{ord}(\Xi_3) &= \frac{1}{2} \cdot 2^8 = 128
\end{aligned}$$

thus $\text{ord}(\Xi) = \sum_{i=0}^{3} \text{ord}(\Xi_i) = 2 \cdot 240 = 480$ which is two times the number of elements of the inner shell of the E_8-lattice.

Interpretations

1. Conjugations

The X-Product can be regarded as a generalization of conjugation:
Conjugation measures commutativity of an algebra, i.e. $\mathbf{AB} = \mathbf{BA}$, so $\mathbf{B} = \mathbf{A}^{-1}\mathbf{BA}$. Analogously the X-product measures associativity of an algebra, i.e. $(\mathbf{AB})\mathbf{X} = \mathbf{A}(\mathbf{BX})$, thus $\mathbf{AB} = (\mathbf{A}(\mathbf{BX}))\mathbf{X}^{-1}$ (due to the *right inverse property* of the octonions).
This means that the X-product "picks" disconnected associative subspaces, which allows them to posses a group structure.
The commutator in the first case is replaced by the associator in the second case which takes its role. The subspace in the second case, i.e. the group need not to be Abelian any more. This means that the second condition in a way generalizes the concept of a **Cartan subalgebra**, which is Abelian. One could therefore speak of a non-Abelian **Cartan subalgebra**. (One might speculate if a further generalization to alternative subspaces makes sense, although they are not a group any more and it is not a priori clear what a root and an eigenvalue means in this context).

2. Integral octonions

Restricting $\mathbf{A}$ and $\mathbf{B}$ to be integral octonions which span the **E_8 root lattice**, one can interpret the X-product as follows:
We start with an integral octonion $\mathbf{A}$ which can be regarded as a vector representing a vertex of the Gosset polytope. Taking any other element $\mathbf{B}$ and multiplying $\mathbf{A}$ with it results in a vector representing one of the 240 vertices of the polytope. The algebra under the (integral-) octonion product is closed. On the other hand we can pick an integral element $\mathbf{X}$ and multiply $\mathbf{A}$ with it in an alternative way, given by one of the other 479 octonion-algebras, i.e. we "step outside" our initial algebra. The importance here is that nevertheless we will not "fall off" the E_8-lattice this way. Multiplying $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{X}$ first and then applying our initial product means that we permute the elements (or equivalently vertices of the polytope) beforehand, but this doesn't interfere with the closure of our initial product applied afterwards because it still acts on the same set of 240 elements.

3. Nucleus

An $\mathbf{X}_i \in \mathbb{O}$ corresponds to an element in the middle-, left- and right-nucleus of the octonion algebra.
Given any elements $\mathbf{A}, \mathbf{B} \in \mathbb{O}$ one has:

- Left nucleus: $(\mathbf{X}^{-1}\mathbf{A})\mathbf{B} = \mathbf{X}^{-1}(\mathbf{AB}) \rightarrow \mathbf{X}((\mathbf{X}^{-1}\mathbf{A})\mathbf{B}) = \mathbf{AB} \equiv \mathbf{A} \circ_{\mathbf{X}} \mathbf{B}$
- Middle nucleus: $(\mathbf{AX})(\mathbf{B}^{-1}\mathbf{X})^{-1} = \mathbf{A}(\mathbf{X}((\mathbf{B}^{-1}\mathbf{X})^{-1}) \rightarrow (\mathbf{AX})(\mathbf{X}^{-1}\mathbf{B}) = \mathbf{A}(\mathbf{X}(\mathbf{X}^{-1}\mathbf{B})) = \mathbf{AB} \equiv \mathbf{A} \circ_{\mathbf{X}} \mathbf{B}$
- Right nucleus: $(\mathbf{AB})\mathbf{X} = \mathbf{A}(\mathbf{BX}) \rightarrow \mathbf{AB} = (\mathbf{A}(\mathbf{BX}))\mathbf{X}^{-1} \equiv \mathbf{A} \circ_{\mathbf{X}} \mathbf{B}$

The set of all elements in the nucleus is a non-Abelian group which in case of the octonions amounts to the **E8**. The nucleus is therefore a 240-dimensional linear space spanned by the 240 basis elements of E_8.

Notice that the octonions have 7 quaternion subalgebras. These are contained in the nucleus. By flipping the signs of the basis elements, each subalgebra contributes with 8 basis elements to E_8, summing up to $7 \cdot 8 = 56$ basis elements.
Furthermore the duals of these algebras are also contained in the nucleus, contributing with another 56 basis elements. *TODO show or check that..*

4. *Multiplication tables*

In the associative case one gets equivalent multiplication tables if one applies an automorphism

$$\phi(\mathbf{E}_i\mathbf{E}_j) = \phi(\mathbf{E}_i)\phi(\mathbf{E}_j)$$

or

$$(R\mathbf{E}_i R^{-1})(R\mathbf{E}_j R^{-1}) = R\mathbf{E}_i R^{-1} R\mathbf{E}_j R^{-1} = R\mathbf{E}_i \mathbf{E}_j R^{-1}$$

In the non-associative case this situation is more restricted. For the octonions, instead of *SO(7)*, one "merely" has **G_2** as automorphisms group.

But there is still the possibility of discrete (parity) transformations. The X-product takes care of that.
Yet for this one must go beyond automorphisms and these transformations complement those that preserve a given multiplication table, given by the action of the **Lie group** G_2.
But we are allowed to have autotopisms in the nonassociative case which can do justice to this more general situation. I.e.

$$(R\mathbf{E}_i R^{-1})(R\mathbf{E}_j R^{-1}) \neq R\mathbf{E}_i R^{-1} R\mathbf{E}_j R^{-1}$$

We rewrite this according to

$$(\mathbf{E}_i' R^{-1})(R\mathbf{E}_j'') = \mathbf{E}_i' \circ_{\mathbf{R}} \mathbf{E}_j''$$

which amounts to the X-product.

Still, transformations represented by the X-product are transformations between equivalent multiplication tables. Although the tables "look" different in that their associated **Fano planes** are different, their associated algebras are isomorphic. These transformations, mapping **Fano planes** from one to another, are given by the action of the group **PSL(2,7)** which is of order 168.
Disregarding different possible signs of base vectors for the moment, out of the $7!$ possible Fano planes and hence multiplication tables one is left with $30 = 7!/168$ inequivalent ones after modding out the equivalent ones by $PSL(7,2)$.

5. Octonionic parity

The X-element of the X-product could be interpreted as a parity. Given an X, the set Ξ_i is fixed, i.e. the parity of the E_8-polytope. Multiplication via the X-product now lets one do algebra on this specific polytope. If one changes the X (which can be compared to changing a left-handed to a right-handed basis in three dimensions, i.e. changing the underlying multiplication table) one changes the parity of the polytope. This can be interpreted as discrete (orthogonal ?) transformations, like reflections.

Papers:

- [1] The Seven-sphere and its Kac-Moody Algebra (1993) - M. Cederwall, C. R. Preitschopf local pct. 47 prl. 9
- Octonion X-Product Orbits (1994) - G. Dixon local pct. 12
- Clifford Algebra-parametrized Octonions and Generalizations (2006) - R. da Rocha, J. Vaz, Jr. local pct. 11
- [2] Octonions: E_8 Lattice to Λ_{16} (1995) - G. Dixon local pct. 9
- Octonion XY-Product (1995) - G. Dixon local pct. 7
- Octonions: Invariant Leech Lattice Exposed (1995) - G. Dixon local pct. 3
- S^7 Current Algebras (1994) - M. Cederwall local pct. 3
- Isotopic Liftings of Clifford Algebras and Applications in Elementary Particle Mass Matrices (2007) - R. da Rocha, J. Vaz Jr. local pct. 2

Links:

- Octonion Products and Lattices - T. Smith

Google Books:

- Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics (1994) - G. M. Dixon local bct. 179

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